Alain Badiou

Number and Numbers

(Des Travaux/Seuil 1990)

Translation © 2005 by Robin Mackay

DRAFT

updated 12 September 2005

Contents

Introduction

0. The Necessity of Thinking Number

0.1. A paradox: we live in the era of the despotism of number, thought is submitted to the law of denumerable multiplicities, and yet (or rather precisely in so far as this default, this failure, is nothing but the obscure obverse of a submission without concept) we have at our disposal no recent, active idea of what number is. The question has been the subject of immense intellectual effort, but for the most part the significant achievements of this labour belong to the beginning of the twentieth century: they are those of Dedekind, Frege, Cantor, and Peano. The factual impact of number brings with it only a silence of the concept. How can we understand today Dedekind's question, posed in his 1888 treatise, *Was sind und was sollen die Zahlen*[1]? What purpose numbers serve, we know very well: they serve, strictly speaking, for everything, they provide the norm for everything. But what they *are* we don't know, or we repeat what the great thinkers of the end of the nineteenth century – no doubt anticipating the extent of their future domain – said they were.

0.2. That number reigns, that the imperative must be: "count!" – who doubts this today? And not in the sense of that maxim which, as Dedekind knew, demands the use of the original Greek when reinscribed[2]:

αει ο ανθροπος αριθμητιζει

– because it prescribes for thought its singular condition in the matheme. But in the factual empire of number, it is not a question of thought. It's a question of realities.

---

[1] [Translated into English as "The Nature and Meaning of Numbers" in *Essays on the Theory of Numbers*, trans. Wooster Woodruff Beman (La Salle, Ill.: Open Court,1901;Reprinted NY:Dover 1963) - trans.].

[2] [*aei o anthropos arithmetizei* - "man always counts". Plutarch (*Convivialium disputationum*, liber 8,2) reports that : "Plato said God geometrizes continually". Kepler's repetition of the statement in his *Mysterium Cosmographicum* (either placatory or ironic, given the decidedly non-platonic nature of his proposed *coelestis machina*) was followed by Gauss's modification: o theos arithmetizei, god arithmetizes, counts or calculates. Dedekind's 'copernican revolution' consisted of transforming this once again into: *aei o anthropos arithmetizei* - man is always counting; completing the transformation

0.3. Firstly, number rules our political conceptions, with the currency (consensual, even if all politics of the thinkable is enfeebled) of suffrage, of opinion polls, of the majority. Every "political" assembly, general or local, municipal or international, voting-booth or public meeting, is settled with a count. And every opinion is measured by the standard of an incessant enumeration of its advocates (even if such an enumeration makes of every fidelity an infidelity). What counts – in the sense of what is valued – is that which is counted. Inversely, everything that deals with numbers must be valued. "Political Science" finesses numbers within numbers, cross-references series of numbers, its only object being *shifts in voting patterns* – that is, changes – usually infinitesimal – in the tabulation of numbers. So political "thought" is a numerical exegesis.

0.4. Number rules over the quasi-totality of the "human sciences" (even if this ciphered alibi can scarcely hide the fact when we speak here of "science", what we have is a technical assemblage whose pragmatic basis is governmental). It is overrun by the statistical data of the entire domain of its disciplines. The bureaucratisation of knowledge is firstly an infinite excrescence of numbering.

At the beginning of the 20th century, sociology was inaugurated in all its ambition – audacity, even – in the will to collapse the image of the communitarian bond into number. It sought to extend to the social body and to representation the Galilean process of formalisation and mathematisation. But ultimately it succumbed to an anarchic development of this programme. It is now replete with pitiful enumerations which serve only to validate the obvious or to establish parliamentary opportunities.

History has imported statistical techniques *en masse*, and is – often, even chiefly, under the alibi of academic Marxism – becoming a diachronic sociology. It has lost that unique quality that had characterised it, since the Greek and Latin historians, as a discipline of thought: its conscious subordination to the political real. Passing through the different phases of reaction to number – economism, sociologism – it does so only to fall into what is the simple inverse: biography, historicising psychologism.

---

into a secular formula emphasising the formidable power of arithmetic. - trans.]

Medicine itself, apart from its wholesale reduction to its scientific Other (molecular biology), is a wild mass of empirical facts, a huge web of blindly-tested numerical correlations.

"Sciences" of men *made into numbers*, to the point of saturation of all possible correspondences between these numbers and *other numbers*, whatever they might be.

0.5. Number governs cultural representation. Certainly, there is television, viewing figures, advertising. But that is not the most important thing. It is in its very essence that the cultural fabric is woven by number alone. A "cultural fact" is a numerical fact. And inversely, whatever produces number can be assigned a cultural place; that which has no number will not have a name either. Art, which has to do with number only insofar as there is a *thinking* of number, is a culturally unpronounceable word.

0.6. Obviously, number governs the economy, and it is there without doubt that we find what Louis Althusser called the "determination in the last instance" of its supremacy. The ideology of modern parliamentary societies, if they have one, is not humanism, the rights of the subject. It is number, the countable, countability. Every citizen is today expected to be cognizant of foreign trade figures, of the flexibility of the exchange rate, of the developments of the stock market. These figures are presented as the real through which other figures are processed: governmental figures, votes and opinion polls. What is called "the situation" is the intersection of economic numericality and the numericality of opinion. France (or any other nation) is representable only in the account books of an import-export business. The only image of a nation resides in the inextricable heap of numbers in which, so it is said, its power is vested, and which one hopes is deemed worthy by those who record its spiritual state.

0.7. Number informs our souls. What is it to exist, if not to assert oneself through a *favourable account*? In America, one starts by saying how much one earns, an identification that has the merit of honesty. Our old country is more cunning. But still, you don't have to look far to discover numerical topics that everyone can identify with. No-one can be presented as an individual without naming that in which

they count, for whom or for what they are really counted. Our soul has the cold transparency of the figures in which it is resolved.

0.8. Marx: "the icy water of egoistic calculation"[3]. And how! To the point where the Ego of egoism is but a numerical web, so that the "egoistic calculation" becomes the cipher of a cipher.

0.9. But we don't know what a number is, so we don't know what we are.

0.10. Must we stick with Frege, Dedekind, Cantor or Peano? Hasn't anything *happened* in the thinking of number? Is there only the exorbitant extent of its social and subjective reign? And what sort of innocent *culpability* can be attributed to these thinkers? To what extent does their idea of number prefigure this anarchic reign? Did they think number, or the future of generalised numericality? Isn't *another idea* of number necessary, in order for us to turn thought back against the despotism of number, in order to subtract the Subject from it? And has mathematics assisted only silently in the comprehensive socialisation of number, of which latter it had previously held a monopoly? This is what I wish to examine.

---

[3] [Marx and Engels, *Communist Manifesto* - trans.]

1. Genealogies: Frege, Dedekind, Peano, Cantor

1. Greek Number and Modern Number

1.1. The Greek thinkers of number related it back to the One, which, as one sees still in Euclid's *Elements*[1], is not considered by them to be a number. Unity is derived from the supra-numeric being of the One. And a number is a collection of unities, an addition. Underlying this conception is a problematic that goes from the Eleatics through to the Neoplatonists, that of the procession of the Multiple from the One. Number is the scheme of this procession.

1.2. The modern ruination of the Greek thinking of number proceeds from three fundamental causes.

The first is the irruption of the problem of the infinite – ineluctable once, with differential calculus, we deal with the reality of series of numbers that cannot be assigned any terminus, although we speak of their limit. How to think the limits of such series *as numbers,* if the latter are articulated only through the concept of a collection of unities? A series tends towards a limit: it is not affected by the addition of its terms, or of its unities. It does not allow itself to be thought as a procession of the One.

The second cause is that, if the entire edifice of number is supported by the being of the One, which is itself beyond being, it is impossible to introduce without some radical subversion that *other* principle – that ontological stopping-point of number – which is zero, or the void. It could be, certainly – and neoplatonist speculation begins with this assumption – that it is the ineffable and architranscendent character of the One which is denoted by zero. But then the problem comes back to numerical one: how to *number* unity, if the One that supports it is void? This problem is so complex that we shall see that it is, even today, the key to a modern thinking of number.

---

[1] Consider, for example the definition of number in Euclid's *Elements* (Book VII, definition 2): "Αριθμος εστιν το εχ μοναδων σογχειμενον τληθος". We might translate as follows: "Number is every multiple composed of unities". The definition of number is specified secondarily, being dependent upon that of unity. But what does definition I, that of unity, say?
"Μονας εστιν, χαθ ην εχαστον των οντων εν λεγεται." So: "Unity is that on the basis of which each being is said to be one" We can see immediately what ontological substructure (that the One can be said of a being in so far as it is) the mathematical definition of number supposes.

The third reason, and the most contemporary one, is the pure and simple dislocation of the idea of the being of the One. We find ourselves under the jurisdiction of an epoch that *obliges* us to hold that being is essentially multiple. In consequence, number cannot proceed from the supposition of a transcendent being of the One.

1.3. The modern thinking of number therefore finds itself compelled to establish a mathematics subtracted from this supposition. In order to achieve this it can take three different approaches.

The approach of Frege, and later of Russell (which we will call, for brevity, the logicist approach), "extirpates" number from the pure consideration of the laws of thought itself. Number, according to this perspective, is a universal trait *of the concept*, deducible from absolutely aboriginal principles (principles without which thought in general would be impossible).

The approach of Peano and Hilbert (let us say the formalist approach) sets out the numerical field as an operative field on the basis of certain singular axioms. This time, number does not assume any particular position with regard to the laws of thought. It is a system of regulated operations, which the axioms of Peano specify by way of a translucid notational practice, entirely transparent to the material gaze. The space of numerical signs is only the most "aboriginal" of mathematics proper (it is preceded only by purely logical calculations). We might say that the concept of number is here entirely mathematised, in the sense that it is conceived as existent only in the course of its usage: the essence of number is calculation.

The approach of Dedekind, of Cantor, then of Zermelo, of von Neumann and Gödel (which we will call the set-theoretical or "platonising" approach) determines number as a particular case of the hierarchy of sets. The support absolutely antecedent to all construction is the empty set, and "at the other end", so to speak, nothing prevents us from examining infinite numbers. The concept of number is thus referred back to a pure ontology of the multiple, whose great Ideas are the classical axioms of set theory. In this context, "being a number" is a *particular predicate*, proceeding from the decision to consider as such certain classes of sets (the ordinals,

or the cardinals, or the elements of the continuum, etc.) with certain distinctive properties. The essence of number is to be a pure multiple endowed with certain properties relating to its internal order. Number *is*, before having any calculation in view (operations will be defined "on" sets of pre-existing numbers). Here, it is a question of an ontologisation of number.

1.4. My own approach will be as follows:

a) The logicist perspective must be abandoned for reasons of internal consistency: it cannot satisfy the prerequisites of thought, and especially of philosophical thought.

b) The axiomatic, or operational, thesis is the most "amenable" to the ideological socialisation of number: it circumscribes the question of a thinking of number *as such* within a context that is ultimately *technical*.

c) The set-theoretical thesis is the strongest. Even so, we must draw far more radical consequences than those that have prevailed up until now. This book tries to follow the thread of these consequences.

1.5. Hence my plan: Examine the theses of Frege, Dedekind, and Peano. Establish myself within the set-theoretical conception. Radicalise it. Demonstrate (a most important point) that, within the framework of this radicalisation, we will rediscover *also* (but not only) "our" familiar numbers: whole numbers, rational numbers, real numbers, all finally thought outside of ordinary operative manipulations, as subspecies of a *unique* concept of number, itself statutorily inscribed in the pure ontology of the multiple.

1.6. As was necessary, mathematics has already proposed this reinterpretation, but only in a recessive corner of itself, blind to the essence of its own thought. It took place with the theory of surreal numbers, invented at the beginning of the seventies by J.H.Conway (*cf. On Numbers and Games,* 1976), taken up firstly by D.E.Knuth (*cf. Surreal Numbers*, 1974), and then by Harry Gonshor in his canonical book (*An*

Introduction to the Theory of Surreal Numbers, 1986). Any interest we have in the technical details will be strictly subordinated here to the matter in hand: establishing a thinking of number that, fixing the latter's status as a form of the thinking of Being, can free us to the extent that an event, always trans-numeric, summons us, whether this event be political, artistic, scientific or amorous. To limit the glory of number to the important, but not exclusive, glory of Being, and to thereby show that what proceeds from an actual event of truth-fidelity can never be, has never been, *counted*.

1.7. None of the modern thinkers of number (I understand by this, I repeat, those who between Bolzano and Gödel tried to fix the idea of number at the juncture of philosophy and logico-mathematics) have been able to offer a *unified* concept. We ordinarily speak of "number" in the context of natural whole numbers, whole "relatives" (positives and negatives), rational numbers (the "fractions"), real numbers (those which number the linear continuum), and finally complex numbers and quaternions. We also speak of number in a more direct set-theoretical sense in designating types of well-orderedness (the ordinals) and pure quantities of indistinct multiples, infinite quantities included (the cardinals). We might expect a concept of number to subsume all of these cases, or at least the more "classical" among them, that is to say the whole natural numbers, the most obvious scheme of discrete enumeration "one by one", and the real numbers, the schema of the continuum. But nothing of the sort exists.

1.8. The Greeks clearly reserved the concept of number for whole numbers, those that were homogenous with their idea of the composition of number on the basis of the One, since only the natural whole numbers can be represented as a collection of unities. To speak of the continuous, they used geometric terms, such as the relation between sizes, or measurements. So that their powerful conception was essentially marked by that division of mathematical disciplines according to which they can treat of either one or the other of what the Greeks held to be the two possible types of object: numbers (from which arithmetic proceeds) and figures (from which, geometry). This division refers, it seems to me, to two ways in which effective, or materialist, thought dialectically effectuates unity: the algebraic way, which works by

composing, relaying, combining elements; and the topological way, which works by perceiving the proximities, the outlines, the approximations, and whose point of departure is not elementary appurtenances but inclusion, the part, the subset[2]. Such a division remains well founded. Even in mathematics itself, Bourbaki's great treatise has as its first pillars, once the general ontological framework of set theory is assumed, "algebraic structures" and "topological structures". And the validity of this arrangement subtends all dialectical thought.

1.9. It is nevertheless clear that in the eighteenth century it was no longer possible to situate increasingly elaborate mathematical concepts exclusively *on one side only of the opposition arithmetic/geometry.* The triple challenge of the infinite, of zero and of the loss of the idea of the One disperses the idea of number, shreds it into a ramified dialectic of geometry and arithmetic, of the topological and the algebraic. Analytic Cartesian geometry proposes a radical subversion of the distinction from the outset, and that which today we call "number theory" had to appeal to the most complex resources of "geometry", in the broadest sense this word had been taken in for decades. The moderns therefore cannot consider the concept of number as the object whose provenance is foundational (the idea of the One) and whose domain is circumscribed (arithmetic). "Number" is said in many senses. But which of these senses constitutes a concept, and allows something singular to be proposed to thought under this name?

1.10. The response to this question, from the thinkers of whom I speak, is altogether ambiguous and displays no kind of consensus. Dedekind, for example, can legitimately be taken for the one – the first – who with the notion of the cut "generated" real numbers from the rationals in a convincing fashion[3]. But when he poses the question: "What are numbers?" he responds with a general theory of ordinals which certainly could provide a foundation for whole numbers in thought, as a particular case, but which could not be applied directly to real numbers. In what sense, in that case, is it legitimate to take the reals for "numbers"? In the same way,

---

[2] On the dialectic – constitutive of materialist thought – between the algebraic and topological orientations, I refer you to my *Theory du Sujet*(Paris: Seuil, 1982) p.231-249
[3] The theme of the cut is covered, in concept and technique, in chapter XV of this book.

Frege, in *The Foundations of Arithmetic*[4], criticises with great acuity all the foregoing definitions (including the Greek definition of number as "collection of unities") and proposes a concept of "cardinal number" which, in effect, subsumes – on condition of various intricate manouevres to which I shall return – the cardinals in the set-theoretical sense, of which the whole natural numbers represent the finite case. But at the same time he excludes the ordinals, to say nothing of the rationals, the reals or the complex numbers. To use one of his favourite expressions, such numbers do not "fall under the [Fregean] concept" of number. Finally it is clear that Peano's axiomatic defines the whole numbers, and them only, as a regulated operative domain. One can certainly define real numbers directly by a special axiomatic (that of a totally ordered field, archimedean and complete). But if the essence of "number" is attained only through the specific nature of the statements which constitute its axiomatics, it is evident that the whole numbers and the reals have nothing in common with each other (as regards their concept) given that if one compares the axiomatic of whole numbers and that of reals, these statements are totally dissimilar.

1.11. All of this takes place just as if, challenged to propose a concept of number that can endure the modern ordeal of the defection of the One, our thinkers conserve the concept in one of its "incarnations" (ordinal, cardinal, whole, real...) without being able to justify the fact that, in every case, some use has to be made of the idea and of the word "number". More specifically, they seem incapable of proposing a unified approach, a common ground, for discrete numeration (the whole numbers), for continuous numeration (the reals), and for "general" or set-theoretical, numeration (ordinals and cardinals). And meanwhile, it is very much the problem of the continuum, of the dialectic of the discrete and the continuous, which, saturating and subverting the antique opposition between arithmetic and geometry, constrains the moderns to rethink the idea of number. From this sole point of view, their work, otherwise admirable, is a failure.

---

[4] [English Translation by J.L.Austin, Evanston: Northwestern University Press; 2nd Edition 1980 - trans.]

1.12. The anarchy thus created (and I cannot take this anarchy in thought to be totally unconnected to the unthinking despotism of number) is that much greater in so far as the methods put in place for each case are totally disparate:

a) The determination of whole numbers can be achieved either by means of a special axiomatic, at whose heart is the principle of recurrence (Peano), or by a particular (finite) case of a theory of ordinals, in which case the principle of recurrence becomes a theorem (Dedekind).

b) To engender the negative numbers, we must introduce algebraic manipulations, which do not bear on the "being" of number, but on its operative dispositions, on structures (symmetricisation of addition).

c) We can repeat these manipulations to obtain the rational numbers (symmetricisation of multiplication).

d) Only a fundamental rupture, which this time falls within the domain of topology, can found the passage to real numbers (consideration of infinite subsets of the set of rationals, cuts or Cauchy series).

e) We return to algebra to construct the field of complex numbers (algebraic enclosure of the field of reals, adjunction of the "ideal" element $i=\sqrt{-1}$, or direct operative axiomatisation on pairs of real numbers).

f) The ordinals are introduced by the consideration of types of order (Cantor), or by the use of the concept of transitivity (von Neumann).

g) The cardinals are restored by a totally different procedure, that of biunivocal correspondence[5].

1.13. How is it possible to extract from this arsenal of procedures – itself deployed historically according to tangled lines whose origins stretch back to the Greeks, to the Arab algebraists, and to the Italian calculists of the Renaissance, to all the founders of modern analysis, to the "structuralists" of modern algebra, and to the set-theoretical creations of Dedekind and of Cantor – a clear and univocal sense of number, that we

can think as a type of being or as an operative concept? The thinkers of number have only in fact been able to demonstrate how the intellectual procedure that conducts us to *each* species of "number" leaves number *per se* languishing in the shadow of its name. They remained distant from that "unique number which cannot be another"[6] whose stellar insurrection was proposed by Mallarmé.

1.14. The question, then, is as follows: is there a concept of number capable of subsuming, in a unique type of being, answering to a uniform procedure, at least the whole natural numbers, the rational numbers, the real numbers and the ordinal numbers, finite or infinite? And does it make sense to speak of a number without knowing how to specify right away to which singular assemblage, irreducible to any other, it belongs? The answer is yes. It is here that we propose the marginal theory, which I wish to make philosophically central, of "surreal numbers".

It is this theory which proposes to us the true contemporary concept of number, and in doing so, it overcomes the impasse of the thinking of number in its modern-classical form, that of Dedekind, of Frege and of Cantor. Upon the basis of it, and as the result of a long labour of thought, we can prevail over the blinding despotism of the numerical unthought.

1.15. We must speak not of one unique age of modern thinking of number, but of what one might call, taking up an expression that Natacha Michel applies to literature, the "first modernity" of the thinking of number[7]. The names of this first modernity are not those of Proust and Joyce, they are those of Bolzano, Frege, Cantor, Dedekind and Peano. I am attempting the passage to a second modernity.

1.16. I have said that the three challenges that a modern doctrine of number must address are those of the infinite, of zero, and of the absence of all foundation on the

---

[5] For a particularly rapid introduction to the different types of numbers which modern analysis uses, refer for example to a book of J Dieudonné, *Elements d'analyse*, t. I, *Fondements de l'analyse moderne*, Paris, Gauthier-Villars, 3e ed., 1981, chap I-IV.

[6] [From *Un Coup De Dés Jamais N'abolira le Hasard*, translated by Brian Coffey as *Dice Thrown*, in Mary Ann Caws (ed.) *Selected Prose and Poetry* (New York: New Directions, 1982) – trans.]

[7] Natacha Michel proposes the distinction between "first modernity" and "second modernity" in a conference paper *L'Instant Persuasif du Roman* (Paris, 1987).

part of the One. If we compare Frege and Dedekind – so close on so many points – on this, it is immediately remarkable that *the order* in which they arrange their responses to these challenges differs essentially:

On the infinite. Dedekind, with admirable profundity, *begins with the infinite*, which he determines with a celebrated positive property: "A system S is said to be infinite when it is similar to a proper part of itself."[8] And he undertakes immediately to "demonstrate" that such an infinite system exists. The finite will be determined afterwards, and it is the finite that is the negation of the infinite (in which regard Dedekind's numerical dialectic has something of the Hegelian about it[9]). Frege, on the other hand, begins with the finite, by means of the whole natural numbers, of which the infinite will be the "prolongation", or the recollection in the concept.

On zero. Dedekind rejects the void and its mark, he says it quite explicitly: "[W]e intend here for certain reasons wholly to exclude the empty system which contains no elements at all."[10] Frege does the contrary by making the statement "zero is a number" the foundation-stone of his whole edifice.

On the One. There is no trace of any privileging of the One in Frege (precisely because he starts audaciously with zero). So one – rather than the One – comes only in second place, as that which falls under the concept 'identical to zero' (the one and only object that falls under the concept being zero itself, we are entitled to say that the extension of the concept is one). On the other hand, Dedekind proposes to retain the idea that we should "begin" with one: "the base-element 1 is called the base-number of the number-series N."[11] And, correlatively, Dedekind doesn't hesitate to fall back on the idea of an absolute All of thought, an idea which Frege's formalism will not let pass as such: "My own realm of thoughts, *i.e.* the totality S of all things, which can be objects of my thought, is infinite."[12] So true is it that in keeping track of the rights of the One, we suppose the All, because the All is that which, necessarily, *proceeds from the One*, as soon as the One is.

---

[8] [Dedekind, *op.cit.*, 64 (references given are to the numbered paragraphs of Dedekind's treatise). –trans.]

[9] I comment in detail on the Hegelian concept of number, which has the virtue that, according to it, the infinite is the truth of the pure *presence* of the finite, in meditation 15 of *L'Etre et l'Evenement* (Paris: Seuil, 1988) p.181-190.

[10] [Dedekind, *op.cit.*, 2. – trans.]

[11] [*Ibid.*, 73 – trans.]

[12] [*Ibid.*, 66 – trans.]

1.17. These divergences of order are not merely technical. They relate, for each of the thinkers, to the centre of gravity of their conception of number and – as we shall see – the stopping-point, at the same time as the foundation, of their thought: the infinite and existence for Dedekind, zero and the concept for Frege.

1.18. The passage to a second modernity of the thinking of number constrains thought to *return* to zero, the infinite, and the One. A total dissipation of the One, an ontological decision as to the being of the void and that which marks it, proliferation without measure of *infinities*: these are the parameters of such a passage. The amputation of the One delivers us to the unicity of the void and to the dissemination of the infinite.

2. Frege

2.1. Frege maintains that number finds its source in pure thought. Like Mallarmé, although without the effects of chance, Frege thinks that "every thought emits a dicethrow"[1]. What is called Frege's "logicism" is most profound: number is not a singular form of being, or a particular property of things. It is neither empirical nor transcendental. It is not a constitutive category either, it is deduced from the concept; it is, in Frege's own words, a *property of the concept*[2].

2.2. The pivotal property that permits the transition from pure concept to number is that of the *extension* of the concept. How is this to be understood? Given any concept whatsoever, an object "falls" under this concept if it is a "true case" of this concept, if the statement that attributes to this object the property comprised in the concept is a true statement. If, in other words, the object validates the concept. Note that everything originates from the truth-value of statements, which is their denotation (truth or falsity). We might hold that if the concept generates number, it does so only in so far as there is truth. Number is in this sense the index of truth, and not the index of being.

But the notion of extension is ramified, obscure.

2.3. Given a concept, by extension of that concept we mean all the truth-cases (every object as truth-case) that fall under this concept. Every concept has an extension.

Now, take two concepts C_1 and C_2. We say that they are *equinumerous*[3] if a biunivocal correspondence exists that associates object for object that which falls under concept C_1 and that which falls under concept C_2. That is, if one can define a biunivocal correspondence between the extension of C_1 and the extension of C_2.

---

[1] [Mallarmé, *op. cit.* (translation modified) – trans.]

[2] The key text for Frege's conception of number is : G.Frege, *Les Fondements de L'arithmétique*, translated from the German by C.Imbert (Paris: Seuil, 1969). [English translation as cited in the note to **1.10.** above - trans.] The first German edition is from 1884. The fundamental argument, extremely dense, occupies paragraphs 55 to 86 (less than thirty pages in the cited edition).
We must salute the excellent work of Claude Imbert, in particular in his lengthy introduction.

[3] [In Austin's translation, *equal* – trans.]

We can see clearly that Frege orients himself within a 'cardinal' definition of number; that he does not care for the structural order of that which falls under the concept. The essential tool that is biunivocity is a characteristic effect of every attempt to 'number' the multiple in itself, the pure multiple subtracted from all structural considerations. To say that two concepts are equinumerous is to say that they have the 'same quantity', that their extensions have the same extent: an abstraction made from all consideration of what the objects *are* that fall under those concepts.

2.4. Number consists in *marking* equinumerosity, the quantitative identity of concepts. Whence the famous definition: "The number that pertains to concept C is the extension of the concept 'equinumerous to concept C'". Which is to say: every concept C generates a number – namely, the set of concepts equinumerous to C, having the 'same pure quantity', the same quantity of extension, as C. Note that a number, grasped in its being, always designates a *set of concepts*, namely all those which validate the statement: 'is a concept equinumerous to C'.

2.5. The sequence through which the concept of number is constructed is as follows:

Concept → Truth → Objects that fall under the concept (that validate the statement of the attribution of the concept to the object) → Extension of the concept (all the truth-cases of the concept) → Equinumerosity of two concepts (via biunivocal correspondence of their extensions) → Concepts that fall under the concept of equinumerosity to concept C (that validate the statement 'is equinumerous to C') → Extension of equinumerosity to C (the set of concepts of the preceding stage) → Number that appertains to concept C (the number is thus the name of the extension of equinumerosity to C).

From a simplified and operative point of view, we can also say that one departs from the concept, that one passes to the object, on condition of truth; that one compares concepts, and that the number names a set of concepts which have in common a property made possible and defined by this comparison (equinumerosity).

2.6. To rediscover the "normal", familiar numbers, on the basis of this pure conceptualism conditioned by truth alone, Frege begins with his admirable deduction of zero: zero is the number belonging to the concept 'not identical to itself'. Since every object is identical to itself, the extension of the concept 'not identical to itself' is empty. Thus zero is the set of concepts whose extension is empty, and which, by virrtue of this fact, are equinumerous with the concept 'not identical to itself'. That is to say precisely that zero is that number which belongs to *every* concept whose extension is empty, void.

I have indicated in **1.17.** the passage to the number 1: "One" is the number that belongs to the concept "identical to zero". It is interesting to note that Frege emphasises, with regard to 1, that it has no "intuitive" or empirical privilege, any more than it is a transcendental foundation: "The definition of 1 does not presuppose, for its objective legitimacy, any matter of observed fact."[4] Without any doubt, Frege participates in the great modern process of the destitution of the One.

The engendering of the series of numbers beyond 1 poses only technical problems, whose resolution, as one passes from n to $n+1$, is to construct between the extensions of corresponding concepts a correlation such that the "remainder" is exactly 1 – which has already been defined.

2.7. Thus the deduction of number as a consequence of the concept appears to have been accomplished. More exactly: from the triplet concept/truth/object, and from that unique formal operator that is biunivocal correspondence, number arises as an instance of pure thought, or integrally logical production; thought *must* suppose itself, in the form of a concept *susceptible* to having truth-cases (and therefore endowed with an extension). This being granted, then thought presupposes number.

2.8. Why choose in particular the concept "not identical to itself" to found zero? One might choose any concept of which it is certain that the extension is empty, of which no thinkable object could have the property it designates. For example "square circle" – a concept which, in fact, Frege declares "is not so black as [it is] painted"[5]. Since it

---

[4] [Frege, *op.cit.* §77 – trans.]
[5] [*ibid.*, §74 – trans.]

is a matter of an integrally conceptual determination of number, this arbitrariness of choice of concept is a little embarrassing. Frege is aware of this, since he writes: "I could have used for the definition of nought any other concept under which no object falls."[6] To obviate his own objection, he invokes Leibniz: the principle of identity, which says that every object is identical to itself, has the merit of being "purely logical"[7]. Purely logical? We understood that it was a matter of legitimating the categories of logico-mathematics (in particular, number) on the sole basis of laws of pure thought. Isn't there a risk of circularity if a logical rule is required right at the outset? We might say then that 'identical to itself' should not be confused with 'equal to itself'. Certainly, equality is one of the logical, or operational, predicates whose foundation is in question (in particular, equality between numbers). But if 'identity' must here be carefully distinguished from the logical predicate of equality, it is clear that the statement 'every object is identical to itself' is not a 'purely logical' statement. *It is an onto-logical statement*. And, as ontological statement, it is immediately disputable: no Hegelian, for example, would admit the universal validity of the principle of identity. For this supposed Hegelian, the extension of the concept "not identical to itself" is anything but empty!

2.9. The purely *a priori* determination of a concept whose extension is certain to be empty is an impossible task without powerful prior ontological axioms. The impasse into which Frege falls is that of an uncontrolled doctrine *of the object*. Because, with regard to the pure concept, what is an "object" in general, an arbitrary object taken from the total Universe of objects? And why is it required of the object that it must be identical to itself, when it is not even required of the concept that it must be non-contradictory to be legitimate, as Frege indicates by his positive regard for concepts of the 'square circle' type, which he emphasises are concepts like any other? Why would the law of being of objects be more stringent than the law of being of concepts? Doubtless it is *if one admits Leibnizian ontology*, for which existent objects obey a *different principle* to thinkable objects, the principle of sufficient reason. It thus appears that the deduction of number on the basis of the concept is not so much universal, or "purely logical", as Leibnizian.

---

[6] [*ibid.*, §74 – trans.]
[7] [*ibid.* – trans.]

2.10. To state as obvious that the extension of a concept is this or that (for example that the extension of the concept "not identical to itself" is empty) is tantamount to supposing that we can move without inconvenience from concept to existence, since the extension of a concept puts into play "objects" which fall under the concept. There is a generalised ontological argument here, and it is this very argument that sustains the deduction of number on the basis of a single concept: number belongs to the concept *through the mediation of thinkable objects that fall under the concept.*

2.11. The principal importance of Russell's paradox, communicated to Frege in 1902, is as a challenge to every pretension to legislate over existence on the basis of the concept alone, and especially over the existence of the extension of concepts. Russell presents a concept (in Frege's sense) – the concept 'being a set that is not a element of itself' – which is certainly a completely proper concept (more so, really, than 'not identical to itself') but one, nonetheless, *with no extension.* It is actually contradictory to suppose that "objects", in the sense of sets that 'fall under this concept', form a set themselves[8]. And if they do not form a set, one cannot define any biunivocal correspondence whatsoever for them. So this "extension" does not support equinumerosity, and consequently *no number* appertains to the concept "set that is not an element of itself".

The advent of a numberless concept ruins Frege's general deduction. And, taking into account the fact that the paradoxical concept in question is wholly ordinary (in fact, all the customary sets that mathematicians use validate this concept: they are not elements of themselves), we might well suspect that there probably exist other concepts to which no number appertains. In fact, it is impossible *a priori* to predict the extent of the disaster. Even the concept "not identical to itself" could well prove not to have any existent extension, which is something entirely different from having an empty extension. Let us add that Russell's paradox is purely logical, that is

---

[8] The letter in which Russell makes known to Frege the paradox that would take the name of its author, a letter written in German, is reproduced in English translation in *From Frege to Gödel,* a collection of texts edited by J. van Heijenoort, Cambridge, Harvard University Press, 4th Edition 1981, p.124. Russell concludes with an informal distinction between 'collection' [or 'set', German 'Menge' – trans.] and 'totality': "From this [the paradox], I conclude that under certain circumstances a definable collection [Menge] does not form a totality."

to say precisely demonstrated: to admit the existence of a set of all those sets that are not members of themselves ruins deductive language by introducing a *formal* contradiction (the equivalent of a proposition and its negation).

2.12. A sort of 'repair' was proposed by Zermelo[9]. He argued that we can conclude from the concept the existence of its extension *on condition that we operate within an already-given existence.* Given a concept C *and a domain of existing objects*, you can say that there exists *in this existing domain* the set of objects that fall under the concept – the extension of the concept. Obviously, since this extension is relative to a domain specified in advance, it does not exist "in itself". This is a major ontological shift: in this new framework it is not possible to move from the concept to existence (and thus to number), but only to an existence in some way cut out of a pre-given existence. You can "separate" in a given domain the objects of this domain that validate the property proposed by the concept. This is why Zermelo's principle, which drastically limits the rights of the concept and of language over existence, is called the axiom of separation. And it does indeed seem that on condition of this axiom, one can guarantee against the inconsistency-effects of paradoxes of the Russell type.

2.13. Russell's paradox is not paradoxical in the slightest. It is a materialist argument, because it *demonstrates* that multiple-being is anterior to the statements that affect it. It is impossible, says the "paradox", to accord to language and to the concept the right to legislate without limit over existence. Even to suppose that there is a transcendental function of language is to suppose an already-available existent, within which the power of this function can but carve out, or delimit, the extensions of the concept.

2.14. Can we, by assuming Zermelo's axiom, save the Fregean construction of number? Everything once again depends on the question of zero. I could proceed

[9] Zermelo develops his set-theoretical axiomatic, including the axiom of separation which remedies the Russell paradox, in a 1908 text written in German. It can be found in English translation in van Heijenhoort's collection, cited in the preceding note. It comes from *Investigations in the Foundations of Set Theory*, and especially its first part, "Fundamental definitions and axioms", p.201-206.

thus: given a delimited domain of objects, whose existence is externally guaranteed, I will call "zero" (or empty set, which is the same thing) that which detaches, or separates, in this domain, the concept "not identical to itself", or any other such concept which I can assure myself no objects of the domain fall under. As it is a question of a limited domain, and not, as in Frege's construction, of "all objects" (a formulation which led to the impasse of a Leibnizian choice without criteria), I have some chance of finding such a concept. If, for example, I take a set of black objects, I will call "zero" that which separates in this set the concept "being white". The rest of the construction follows.

2.15. But what domain of objects can I begin with, for which it can be guaranteed that they arise from pure thought, that they are "purely logical"? Frege knew well enough to construct a concept of number that was, according to his own expression, "neither a sensible being, nor a property of external things", and he stresses on several occasions that number is subtracted from the representable. Establishing that number is a production of thought, deducing it from only the abstract attributes of the concept in general – this cannot be achieved using black and white objects. The question then becomes: what existent can I assure myself of, outside of any experience? Is the axiom "something exists" an axiom of pure thought, and, supposing that it is, what property can I discern of which it is *certain* that it does not appertain to any part of this existent "something"?

2.16. A "purely logical" demonstration of existence for the thought of an arbitrary object, of a point of being, of an "object = x", the statement "every x is equal to itself" is an axiom of logic with equality. Now, the universal rules of first-order logic, logic valid for every domain of objects, permit us to deduce from the statement "every x is equal to x" the statement "there exists an x that is equal to x" (subordination of the existential quantifier to the universal quantifier)[10]. Thus x exists (that is to say, at least that x that is equal to itself).

---

[10] The subordination of the existential quantifier to the universal quantifier means that given a property P, if every possible x possesses this property then there exists an x which possesses it. In the predicate calculus: $\forall x(P(x)) \rightarrow \exists x(P(x))$. The classical rules and axioms of predicate calculus permit one to deduce this implication. *Cf.* for example E. Mendelson's manual, *Introduction to Mathematical Logic*, (NY: van Nostrand, 1964) p. 70-71.

Thus we can demonstrate within the framework of set theory, in a purely logical way, *first of all* that a set exists. And *then* we can separate the empty set, within that existent whose existence has been proved, by utilising a property that no element can validate (for example, "not being equal to itself"). We have respected Zermelo's axiom, since we have operated in a pre-given existent, and yet we have succeeded in engendering zero.

2.17. It is I believe quite evident that this "demonstration" is an unconvincing artifice, a logical sleight of hand. From the universal presumption of equality-to-self (which we might possibly accept as an abstract law, or a law of the concept), who could reasonably infer that there *exists* something rather than nothing? If the universe were absolutely void, it would remain logically admissible that, supposing that something existed (which would not be the case), it would be constrained to be equal to itself. The statement "every x is equal to x" would be valid, but there would be no x, so the statement "there exists an x equal to itself" would not be valid.

The passage from a universal statement to an assertion of existence is an exorbitant right which the concept cannot arrogate to itself. It is not possible to establish existence on the basis of a universal law that could be sustained just as well in absolute nothingness (consider for example the statement 'nothing is identical to itself'). And, since no existent object is deducible from pure thought, you cannot distinguish zero therein. Zermelo does not save Frege.

2.18. The existence of zero, or the empty set, and therefore the existence of numbers, is in no wise deducible from the concept, or from language. "Zero exists" is inevitably a *primary* assertion; one, even, that fixes an existence from which all others will proceed. Far from Zermelo's axiom, combined with Frege's logicism, allowing us to engender zero and then the sequence of numbers, it is on the contrary the absolutely inaugural existence of zero (as empty set) that ensures the possibility of separating any extension of a concept whatsoever. Number is primary here: it is the *point of being* upon which the exercise of the concept depends. Number, as number of nothing, or zero, sutures every text to its latent being. The void is not a production of thought, because it is from its existence that thought proceeds, in as much as "it is

the same thing to think and to be"[11]. In this sense, it is the concept that comes from number, and not the other way around.

2.19. Frege's initiative is in certain regards unique: it is not a question of creating new intra-mathematical concepts (as Dedekind and Cantor will do), but of elucidating – with the sole resource of rigorous analysis – what, among the possible objects of thought, distinguishes those which fall under the concept of number. In this sense, my own work follows along the same lines. Its novelty lies only in removing the obstacles by reframing the investigation according to new parameters. Above all, it is a question of showing that thought is not constituted by concepts and statements alone, but also by decisions which engage it within the *epoch* of its exercise.

---

[11] [το γαρ αυτο νοειν εστιν τε και ειναι – From Parmenides' Poem. – trans.]

3. Additional note on a contemporary usage of Frege

3.1. Jacques-Alain Miller, in a 1965 lecture entitled "Suture" and subtitled "Elements of the Logic of the Signifier"[1], proposed a reprise of Frege's construction of number. This text founds a certain logic of compatibility between structuralism and the Lacanian theory of the subject. I myself have periodically returned to this foundation[2], albeit at the cost of raising various problems within it. Twenty-five years later, "I am here; I am still here"[3].

3.2. The question Miller addresses to Frege is the following: "*What is it* that functions in the series of whole natural numbers?". And the response to this question – a response, might I say, wrested by force out of Frege – is that "in the process of the constitution of the series, *the function of the subject*, misrecognised [méconnue], is at work".

3.3. If we take this response seriously, it means that in the last instance, in the proper mode of its misrecognition, it is the function of that subject whose concept Lacan's teaching transmits to us that constitutes, if not the essence, at least the process of engenderment (the "genesis of the progression", says Miller) of number.

Evidently we cannot ignore such a radical thesis. Radical with regard to a *prima facie* reading of Frege's doctrine, which dedicates a specific argument to the refutation of the idea according to which number would be 'subjective'[4] (although it is true that for Frege, 'subjective' means 'caught up in representation', which obviously

---

[1] Miller's text appears in *Cahiers pour L'analyse*, no 1, Paris, Ed. du Seuil, February 1966. One ought to complement its reading by thatof the article in the same number of the review by Y. Duroux, "Psychologie et logique", which examines in detail the successor function in Frege.

[2] *Cf.* A. Badiou "Marque et Manque : a propos du Zéro", in *Cahiers pour L'analyse*, no 10, Paris, ed du Seuil, march 1969.

[3] [*J'y suis, j'y suis toujours.* From Rimbaud's 1872 poem *Qu'est-ce pour nous, mon cœur, que les nappes de sang.* See *Collected Poems*, ed., trans. Oliver Bernard . (London: Penguin, 1986) p202-3. – trans.]

[4] [See Frege, *op.cit*., §26 – trans]

does not cover the Lacanian function of the subject). Radical with regard to my own thesis, since I hold that number is a form of being, and that, far from its being subtended by the function of the subject, it is on the contrary on the basis of number, and especially of that first number-being that is the void (or zero), that the function of the subject receives its share of being.

3.4. It is not a question here of examining what this text – the first important Lacanian text not to be written by Lacan himself – brings to the doctrine of the signifier, or by what analogy it illuminates the importance – at the time still little appreciated – of all that the master taught as to the grasping of the subject through the effects of a chain. It is exclusively a question of examining what Miller's text assumes and proposes *with regard to the thinking of number as such.*

3.5. Miller organises his demonstration as follows:

– To found zero, Frege (as we saw in **2.6.**) invokes the concept "not identical to itself". No object falls under this concept. On this point, Miller emphasises – compounds, even – Frege's Leibnizian reference. To suppose that an object could be not identical to itself, or that it could be non-substitutable for itself, would be entirely to subvert truth. A statement bearing on object A must suppose, in order to be true, the invariance of A in each occurrence of the statement, or "each time" the statement is made. The principle "A is A" is a law of any possible truth. Reciprocally, in order to salvage truth, it is crucial that no object fall under the concept "not identical to itself". Whence zero, which numbers the extension of such a concept.

– Number is thus shown to issue from one single concept, on condition of truth. But this demonstration is consistent only *because in thinking it, one has been able to call upon an object non-identical to itself*, albeit only to discharge it in the inscription of zero. Thus, writes Miller, "the 0 that is inscribed in the place of number consummates the exclusion of this object".

To say that "no object" falls under the concept "not identical to itself" is to cancel out this object, as soon as calling upon it, in this nothing whose only subsisting trace will be, precisely, the mark zero: "Our purpose," Miller concludes, "has been to recognise in the zero-number the suturing placeholder of lack."

– *What is it* that comes to lack thus? What "object" can have as a placeholder for its own absence the first numerical mark; and sustain, with regard to the entire sequence of numbers, the uninscribable place of that which pertains to its vanishing? What is it that insists *between numbers*? We must certainly agree that no "object" can, even by default, fall in that empty place that assigns non-identity-with-self. But there does exist (or here, more precisely, ek-sist) something which is not even object, the *proper sense* of non-object, the object as impossibility of the object: the subject. "The impossible object, which the discourse of logic summons as the not-identical-to-itself and then rejects, *wanting to know nothing of it*, we shall give its name, in so far as it functions as the excess at work in the series of numbers: the subject."

3.6. One must carefully distinguish between what Miller assumes of Frege and what can really be attributed to Frege. I will proceed in three stages.

3.7. *FIRST STAGE.* Miller takes as his point of departure the Leibniz-Frege proposition according to which the "salvation of truth" demands that all objects should be identical to themselves. Here is surreptitiously assumed, in fact, the whole formalisation of the real towards which Leibniz worked all his life, and to which Frege's ideography is the undoubted heir. In this regard, Miller is even right to equate, along with Leibniz, "identical to itself" and "substitutable", thus denoting an equation of object with letter. For what could it mean to speak of the substitutability of an object? Only the letter is entirely substitutable for itself. "A is A" is a principle of letters, not of objects. To be identifiable at one remove from itself, and amenable to questions of substitutability, the object must be under the authority of a letter, which alone renders it over to calculation. If A is not identical at all moments to A, truth (or rather veridicality) *as calculation* is annihilated.

The latent hypothesis is therefore that truth *is of the order of calculation*. It is only given this supposition that, firstly, the object must be represented as a letter; and secondly, the non-identical status of the object-letter can radically subvert truth. And, if truth is of the order of calculation, then zero – which numbers the exclusion of the non-identical-to-itself (the subject) – is itself nothing but a letter, the letter 0. It is straightforward to conclude from this that zero is the inert placeholder of lack, and that what "propels" the series of numbers as an engenderment of marks, a repetition in which is expressed the misrecognition of that which insists, is the function of the subject.

More simply, if truth is saved only by maintaining the principle of identity, then the object emerges in the field of truth only as a letter amenable to calculation. And, if this is the case, number can be sustained only as the repetition of that which insists as lack, which is necessarily the non-object (or the non-letter, which is the same thing), the place where "nothing can be written"[5] – in short, the subject.

3.8. Nothing is retained of Leibnizian being, although he fails to recognise in this philosophy the archetype of one of the three great orientations in thought, the constructivist or nominalist orientation (the other two being the transcendent and the generic[6]). Adopting the generic orientation, I declare that, for truth to be saved, it is necessary precisely to abolish the two great maxims of Leibnizian thought, the principle of non-contradiction and the principle of indiscernibles.

3.9. A truth presupposes that the situation of which it is true occurs as non-identical to itself: this non-identity-with-self is an indication that the situation has been supplemented by a multiple "in excess", whose membership or non-membership of the situation, however, is intrinsically indecidable. I have named this supplement "event", and it is always from an event that a truth-process originates. Now, as soon

---

[5] [ ]

[6] On the typology of orientations in thought, *cf.* meditation 27 in *L'Etre et L'Evenement*, *op. cit.*, p 311-315

as the undecidable event has to be decided, the situation necessarily enters into the vacillation of its identity.

3.10. The process of a truth – puncturing the strata of knowledge to which the situation clings – inscribes itself as indiscernible infinity, that which no established thesaurus of language can designate.

It is enough to say that zero, or the void, has nothing in itself to do with the salvation of truth, that which is at play in the correlative "work" between the undecidability of the event and the indiscernibility of that which results in the situation. No more than it is possible to refer truth to the power of the letter, since the existence of a truth is precisely that to which no inscription can attest. The statement "truth is" – far from wagering that no object falls under the concept of "not identical to itself", and that therefore zero is the number of that concept – instead permits this triple conclusion:

–   There exists an object that occurs as "non-identical-to-itself" (undecidability of the event).

–   There exist an infinity of objects which do not fall under any concept (indiscernibility of a truth).

–   Number is not a category of truth.

3.11. *SECOND STAGE.* What is the strategy of Miller's text? And what role does number, as such, play within it? Is it really a matter of maintaining that the function of the subject is implicated – in the form of a misrecognized foundation – in the essence of number? This is undoubtedly what is stated in all clarity by the formula that I have already cited above: "In the process of the constitution of the series [of numbers] […] the function of the subject [...] is at work." More precisely, only the function of the subject – that which zero, as number, marks in the place of lack, holding the place of its revocation – is capable of explicating what, in the series of numbers, functions as iteration or repetition: being excluded, the subject (the "non-

identical-to-itself") is included through the very insistence of marks, incessantly repeating "one more step", firstly from 0 to 1 ("the 0 counts for 1", notes Miller), then indefinitely, from *n* to *n*+1: "its [the subject's – in the Lacanian sense] *exclusion* from the field of number identifies itself in repetition."

3.12. Other passages of Miller's text are more equivocal, indicating an analogical reading. For example: "If the series of numbers, metonymy of zero, begins with metaphor, if the zero member of the sequence as number is only the suturing placeholder of absence (of absolute zero) which moves beneath the chain according to the alternating movements of a representation and of an exclusion – then what prevents us from recognising, in the restored relation of zero to the series of numbers, the most elementary articulation of the relation of the signifying chain to the subject?" The word "recognising" is compatible with the idea that the Fregean doctrine of number proposes a "matrix" (the title of another article by Miller on the same question[7]) – that is isomorphic (maximum case) or similar (minimum case), but in any case not identical – to the relation of the subject to the signifying chain. Frege's doctrine would then be a pertinent *analogon* of Lacanian logic: To which we would have no reply, since in that case Miller's text *would not be a text on number*. It would be doubly not so: firstly because it speaks not of number, but of Frege's doctrine of number (without taking a position on the validity or the consistency of that doctrine); and secondly because it would present the series of numbers as a didactic vector for the logic of the signifier, and not as an effective example of an implication of the function of the subject in the series of numbers.

3.13. This critical evasion assumes that two conditions are met: that there should be, between the doctrine of number and that of the signifier, isomorphism or similarity, and not identity or exemplification; and that Miller does not account for the validity of the Fregean doctrine of number.

---

[7] "Matrice" in *Ornicar?* 4 (1975). [Trans. Daniel G. Collins in *lacanian ink* 12 (Fall, 1997): 45-51. – trans.]

3.14. On this last point, in which, to my eyes (that is, to one who is concerned with the thinking of number as such) everything hangs in the balance, Miller maintains the suspense at every step. He speaks of " Frege's System" without one's being able to decide whether, yes or no, in his opinion, it's a matter of an actual accomplished theory of number, a theory entirely defensible in its essence. It is striking that at no point in this very subtle and intricate exercise are the immanent difficulties of "Frege's system" ever mentioned – particularly those that I highlighted with regard to zero, the impact of Russell's paradox, Zermelo's axiom and, ultimately, the relation between language and existence. It thus remains *possible* to believe that the signifier/number isomorphism operates between, on the one hand, Lacan, and on the other hand a Frege reduced to a singular theory, whose inconsistency has no impact with regard to the analogical goals pursued.

3.15. Evidently, it remains to be seen whether or not this inconsistency can, as a result, be found *transferred to the other pole of the analogy*, that is to the logic of the signifier. The risk is not inconsequential, when one considers that this latter is placed by Miller in a founding position with regard to logic *tout court*, including, one presumes, Frege's doctrine: "The first (the logic of the signifier) treats of the emergence of the other (the logic of logicians), and should be conceived of as a logic of the origin of logic." But what happens if this process of origination is completed through the theme of the subject by a schema (Frege's) marred by inconsistency? But this is not my problem. Given the conditions of which I have spoken, if the text does not concern itself with number, we are finished here.

3.16. *THIRD STAGE.* There remains, however, an incontestable degree of adherence on Miller's part to a general representation of number, one in which it is conceived of as, in some way, intuitive, and which I cannot accept. It concerns the idea – central, since it is precisely here that the subject makes itself known as the cause of repetition – according to which number is grasped as a "functioning", or in the "genesis of a

progression". This is the image of a number that is "constructed" iteratively, on the basis of that point of puncture that is denoted by zero. This dynamical theme, which would have us see number as passage, as self-production, as engenderment, is omnipresent in Miller's text. The analysis centres precisely on the "passage" from 0 to 1, or on the "paradox of engenderment" of *n*+1 from *n*.

3.17. This image of number as iteration and passage decides in advance any methodical discussion about the essence of number. Even if we can only *traverse* the numeric domain according to some *laws* of progression, of which succession is the most common (but not the only one, far from it), why must it follow that these laws are constitutive of the being of number? It is easy to see why *we* have to "pass" from one number to the next, or from a sequence of numbers to its limit. But it is, to say the least, imprudent thereby to conclude that *number is defined or constituted by such passage*. It might well be (and this is my thesis) that number *does not pass*, that it is immemorially deployed in a swarming coextensive to its being. And we will see that, just as these laborious passages give the rule only to *our* passage through this deployment, it is likely that we remain ignorant of, have at the present time no use for, or no access to, the greater part of what our thought can conceive of as *existent* numbers.

3.18. The "constructivist" thesis that makes of iteration, of succession, of passage, the essence of number, leads to the conclusion that *very few* numbers exist, since here "exist" has no sense apart from that effectively supported by some such passage. Certainly, the intuitionists assume this impoverished perspective. Even a demi-intuitionist like Borel[8] thinks that the great majority of whole natural numbers "don't exist" except as a fictional and inaccessible mass. It could even be that the Leibnizian choice that Miller borrows from Frege is doubled by a latent intuitionist choice.

---

[8] For example E.Borel, "La Philosophie mathematique et l'infini", in *Revue du mois*, no 14, 1912, p 219-227

We must recognize that the logic of the signifier and intuitionist logic have more than a little in common, if only because the latter expressly summons the subject (the "mathematician subject") in its machinery. But in my opinion such a choice would represent an additional reason not to enter into a doctrine of number whose overall effect is that the site of number, measured by the operational intuition of a subject, is inexorably finite. For the domain of number is rather an ontological prescription incommensurable to any subject, and immersed in the infinity of infinities.

3.19. Thus the problem becomes: how to think number whilst admitting, against Leibniz, that there are real indiscernibles; against the intuitionists, that number persists and does not pass; and against the foundational use of the subjective theme, that number exceeds all finitude?

4. Dedekind

4.1. Dedekind[1] introduces his concept of number in the framework of what we would today call a "naive" theory of sets. "Naive" because it concerns a theory of multiplicities that recapitulates various presuppositions about things and thought. "Naive" meaning, in fact: philosophical.

Dedekind states explicitly, in the opening of his text *The Nature and Meaning of Numbers*, that he understands "by *thing* every object of our thought"[2]; and, a little later, that when different things are "for some reason considered from a common point of view, associated in the mind, we say that they form a system S"[3]. A system in Dedekind's sense is therefore quite simply a set in Cantor's sense. The context for Dedekind's work is not the concept (as with Frege), but directly the pure multiple, a collection that counts for one (*a* system) the objects of thought.

4.2. Dedekind develops a conception of number that is essentially *ordinal* (like those of Cantor). We have seen (*cf.* **2.3.**) that Frege's conception was essentially *cardinal* (via the biunivocal correspondences between extensions of concepts). What significance does this opposition have? In the ordinal view, number is thought as the link of a chain, it is an element of a total order. In the cardinal view, it is rather the mark of a "pure quantity" obtained through the abstraction of domains of objects having "the same quantity". The ordinal number is thought according to the schema of a *series*, the cardinal number to that of a *measure*.

4.3. Dedekind affirms that infinite number (the totality of whole numbers, for example) *precedes*, in construction, finite number (each whole, its successor, etc.). It is thus that the existence of an infinite (indeterminate) system, then the particular existence of N (the set of whole natural numbers) forms the contents of paragraphs 66

---

[1] The reference text for Dedekind's doctrine of number is : *Les Nombres, que sont-ils et à quoi servent-ils?*, translated from the German by J.Milner and H.Sinaceur, Paris, Navarin, 179. The first German edition was published in 1888. [English translation as cited in note 1 to **0.1.** above – trans.]
[2] [Dedekind, *op.cit.*, 1 – trans.]
[3] [*ibid.* 2 – trans.]

and 72 in Dedekind's text (in his numbering), whereas a result as apparently elementary as "every number n is different from the following number n' ", comes in paragraph 81.

Dedekind is a true modern. He knows that the infinite is *simpler* than the finite, that it is the most general attribute of being, an intuition from which Pascal – without doubt the first – had drawn radical consequences as regards the place of the subject.

4.4. Dedekind asks first of all that we allow him the philosophical concept of "system", or multiplicity of anything whatsoever (*cf.* **4.1.**). The principal operator will then be, as with Frege (*cf.* **2.3.**), the idea of biunivocal correspondence between two systems. Dedekind, however, will make a use of it totally different to that of Frege.

Let us note in passing that biunivocal correspondence, bijection, is the key notion of all the thinkers of number of this epoch. It organises Frege's thought, Cantor's, and Dedekind's.

4.5. Dedekind calls the function, or correspondence, a "transformation"[4], and that which we call a bijective function or a biunivocal correspondence a "similar transformation"[5]. In any case it is a question of a function f which makes every element of a set (or system) S' correspond to an element (and one only) of a set S, in such a fashion that:

- To two different elements s_1 and s_2 of S will correspond two different elements $f(s_1)$ and $f(s_2)$ of S';

- Every element of S' is the correspondent, through f, of an element of S.

We call a *distinct* (today we would say injective) function, a function that complies only with the first condition:

$$[(s_1 \neq s_2) \rightarrow (f(s_1) \neq f(s_2))]$$

---

[4] [Dedekind, *op.cit.* 21-25 – trans.]
[5] [Dedekind, *op.cit.*, 26-35 – trans.]

We can obviously consider functions *f* defined 'in' a system S, rather than 'between' a system S and another system S'. Functions (or transformations) of this type make every element of S correspond to an element of S (another, or the same one: the function could be the function of identity, at least for the element considered).

4.6. Take, then, a system S, an application *f* (not necessarily one of likeness or a biunivocal one) of S to itself, and s an element of S. We will call the *chain* of the element *s* for the application *f* the set of values of the function obtained in iterating it starting from *s*. The chain of *s* for *f* is then the set whose elements are: *s*, *f(s)*, *f(f(s))*, *f(f(f(s)))*,..., etc.

Do not think that we are dealing here with an infinite iteration: it could very well be that at a certain stage, the values obtained would repeat themselves. This is obviously the case if S is finite, since the possible values, which are the elements of S (the application *f* operates from S to within S), are exhausted after a finite number of stages. But it is also the case when one comes across a value *p* of the function *f* where, for *p*, *f* is identical. Because then *f(p) = p*, and therefore *f(f(p)) = f(p) = p*. The function *halts* at *p*.

4.7. We will say that a system N is (this is Dedekind's expression) *simply infinite*[6] if there exists a transformation *f* of N within N that complies with the three following conditions:

1) The application *f* of N within N is a distinct application (*cf.* **4.5.**).

2) N is the chain of one of its elements, which latter Dedekind denotes as 1, and which he calls the *base-element* of N.

3) The base-element 1 is not the correspondent through *f* of any element of N. In other words, for any *n* which is part of N, *f(n) ≠ 1* : the function *f* never "returns" to 1.

---

[6] [see Dedekind, *op.cit.*, 71 – trans.]

We can make a rather simple demonstration of such an N. We "begin" with the element 1. We know (condition 3) that $f(1)$ is an element of N different from 1. We see next that $f(f(1))$ is different to 1 (which is never a value for f). But $f(f(1))$, equally, is different to $f(1)$. In fact, the function f (condition 1) is a distinct application – so two different elements must correspond through f to different elements. From the fact that 1 is different from $f(1)$ it follows in consequence that $f(1)$ is different from $f(f(1))$. More generally, every element thus obtained by the iteration of the function f will be different to all those that 'preceded' it. And since N (condition 2) is nothing other than the chain thus formed, N will be composed of an 'infinity' (in the intuitive sense) of elements, all different, ordered by the function f in the sense that each element 'emergeS' via an additional step of the process that begins with 1 and is pursued by continuously applying operation f.

4.8. The 'system' N thus defined is the *site of number*. Why? Because all the usual "numerical" manipulations can be defined on the elements n of such a set N.

By virtue of the function f, we can pass without difficulty on to the concept of 'successor' of a number: if n is a number, $f(n)$ is its successor. It is here that Dedekind's 'ordinal' orientation comes into effect: the function f, via the mediation of the concept of the chain, is that which defines N as the space of a total order. The first "point" of this order is obviously 1. For philosophical reasons (*cf*. **1.17.**), Dedekind prefers a denotation beginning from 1 to one beginning from 0; "1" denotes in effect the first link of a chain, whereas zero is "cardinal" in its very being: it marks lack, the class of all empty extensions.

With 1 and the operation of succession, we can without difficulty obtain, first the primitive theorems concerning the structure of the order of numbers, and then the definition of arithmetical operations, addition and multiplication. We will have rediscovered, on the sole basis of the concepts of 'system' (or set) and of 'similar function' (or biunivocal correspondence), the 'natural' domain of numericality.

4.9. A system N, structured by a function *f* which complies with the three conditions above (**4.7.**), will be called "a system of numbers", a site of the set of numbers. To cite Dedekind[7]:

> If, in the consideration of a simply infinite system N, set in order by a transformation *f*, we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation *f* then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the base-number of the *number-series* N. With reference to this freeing the elements from every other content (abstraction), we are justified in calling numbers a free creation of the human mind.

The enthusiastic tone leaves no room for doubt. Dedekind is conscious of having, with his purely functional and ordinal engenderment of the system S, torn number away from every form of external jurisdiction, in the direction of pure thought. This was already the tone, and these the stakes, of the 'proclamation' which appeared in the preface to the first edition of his brochure: "In speaking of arithmetic (algebra, analysis) as a part of logic, I mean to imply that I consider the number concept to be entirely independent of the notions or intuitions of space and time, that I consider it more as an immediate result from of the laws of thought." This is a text that, as will be appreciated, lends itself to a Kantian interpretation: the whole problem of modern thinkers of number is to navigate within the triangle Plato-Kant-Leibniz[8]. In defining, not 'a' number, but N, the simply infinite 'system' of numbers, Dedekind considers, with legitimate pride, that he has established himself, by means of the power of thought alone, in the intelligible site of numericality.

4.10. Informed by Frege's difficulties, which do not concern his concept of zero and of number, but the transition from concept to existence or the jurisdiction of language

---

[7] [Dedekind, *op.cit.*, 73. Dedekind's text has ɸ where Badiou uses *f* – trans.]

[8] One can hold that Frege is a Leibnizian, that Peano is a Kantian, and that Cantor is a Platonician.
 The greatest logician of our times, Kurt Gödel, considered that the three most important philosophers were Plato, Leibniz and Husserl, this last, if one might say so, holding the place of Kant.
 The three great questions which mathematics poses were thus:
   a) The reality of the pure intelligible, the being of that which mathematics thinks (Plato).
   b) The development of a well-formed language, the certitude of inference, the law of calculation (Leibniz).

over being, we ask: does a system of numbers, a "simply infinite" system N, *exist*? Or will some unsuspected "paradoxes" come to temper, for us, Dedekind's intellectual enthusiasm?

4.11. Dedekind is evidently concerned about the existence of his system of number. In order to establish it, he proceeds in three steps:

1) Intrinsic definition, with no recourse to philosophy or to intuition, of what an infinite system (or set) is.

2) Demonstration (this, as we shall see, highly speculative) of the existence of an infinite system.

3) Demonstration of the fact that all infinite systems "contain as a proper part a simply infinite system N".

These three points permit the following conclusion: since there exists at least one infinite system, and every infinite system has as subsystem an N, a simply infinite system or "site of number", this site exists. Which is to say: number exists. The idea that "arithmetic should be a part of logic" means that, by means of the exclusively conceptual work of pure thought, I can guarantee the consistency of an intelligible site of numericality, and the effective existence of such a site.

4.12. The definition of an infinite set that Dedekind proposes is remarkable. He was very proud of it himself, with good reason. He notes that "the definition of the infinite (...) forms the core of my whole investigation. All other attempts that have come to my knowledge to distinguish the infinite from the finite seem to me to have met with so little success that I think I may be permitted to forego any critique of them."[9]

This definition of the infinite systematises a remark already made by Galileo: there is a biunivocal correspondence between the whole numbers and the numbers that are their squares. Suffice it to say that $f(n) = n^2$. However, the square numbers constitute a proper part of the whole numbers (one calls a proper part of a set a part

---

c)The constitution of sense, the universality of statements (Kant, Husserl).

which is different from the whole, a truly "partial" part). It seems, therefore, that if one examines intuitively infinite sets, there exist biunivocal correspondences between the sets as a whole and one of their proper parts. This part, then, has "as many" elements as the set itself. Galileo concluded that it was absurd to try and think of actual infinite sets. Since an infinite set is "as large" (contains "as many" elements) as one of its proper parts, the statement "the whole is greater than the part" is apparently *false* when one considers infinite totalities. Now, this statement is an axiom of Euclid's *Elements*, and Galileo did not think it could be renounced.

Dedekind audaciously transforms this paradox into the *definition* of infinite sets: "A system S is said to be *infinite* when it is similar to a proper part of itself. In the contrary case, S is said to be a *finite* system."[10] (Remember that, in Dedekind's terminology, 'system' means set, and the similarity of two systems means that a biunivocal correspondence exists between them.)

4.13. The most striking aspect of Dedekind's definition is that it determines infinity *positively*, and subordinates the finite negatively. This is its especially modern accent, something that one almost always finds in Dedekind. An infinite system has a property of an existential nature: there *exists* a biunivocal correspondence between it and one of its proper parts. The finite is that for which such a property *does not obtain*. The finite is simply that which is not infinite, and all the positive simplicity of thought directs itself to the infinite. This intrepid total secularisation of the infinite is a gesture whose virtues we (clumsy disciples of "finitude", in which our religious dependence still lies) have not yet exhausted.

4.14. The *third* point of Dedekind's approach (that every infinite system contains as one of its parts a system of type N, a site of number, *cf*. **4.11.**) is a perfectly elegant proof.

Suppose that a system S is infinite. Then, given the definition of infinite systems, there exists a biunivocal correspondence *f* between S and one of its proper

---

[9] [Dedekind, *op.cit.*, 64n. – trans.]
[10] [*ibid.* – trans.]

parts S'. In other words a bijective function *f* that makes every element of S correspond to an element of S'. Since S' is a *proper* part of S, there is at least one element of S that is not in the part S' (otherwise one would have S= S', and S' would be "not proper"). We choose such an element, and call it 1. Consider the chain of 1 for the function *f* (for 'chain' *cf*. **4.6.**). We know that:

-   *f* is an distinct (injective) transformation, or function, since it is precisely the biunivocal correspondence between S and S', and all biunivocal correspondence is distinct.

-   1 certainly does not correspond, through *f*, to any other term of the chain, since we have chosen 1 outside of S', and *f only makes elements of S' correspond* to elements of S. An element *s* such that *f(s)* = 1 therefore cannot exist in the chain. In the chain, the function never 'returns' to 1.

The chain of 1 for *f* in S is, then, a simply infinite set N: it complies with the three conditions set for such an N in paragraph **4.7**.

We are thereby assured that, *if an infinite system exists*, then an N, a site of number, also exists as part of S. Dedekind's thesis is ultimately the following: *if the infinite exists, number exists*. This point (taking account of the ordinal definition of number as the chain of 1 for a similar transformation, and of the definition of the infinite) is exactly *proved*.

4.15. But does the infinite exist? There lies the whole question. This is point two of Dedekind's approach, where we see that *the infinite, upon which the existence of number relies, occupies in Dedekind the place that is occupied by zero in Frege*.

4.16. To construct the proof of that upon which everything will rest from now on (the consistency in thought, and the existence, of an infinite system or set), Dedekind hastily canvasses all of his initial philosophical presuppositions (the thing as object of thought). Of course, these presuppositions already gently sustain the very idea of a "system" (collection of any things whatsoever). But we have had the time, seized by the superb smooth surface of the subsequent definitions (chain, simply infinite set) and the proofs, to forget this fragility. One cannot do better than to cite here the

"proof" of what is offered blithely, in Dedekind's text, as the "theorem" of paragraph 66:

> 66. *Theorem:* There exist infinite systems.
>
> *Demonstration*: My own realm of thoughts, *i.e.* the totality S of all the things, which can be objects of my thought, is infinite. For, if *s* signifies an element of S, then is the thought *s'*, that *s* can be an object of my thought, itself an element of S. If we regard this as transformation *f(s)* of the element *s*, then has the transformation *f* of S thus determined the property that the transformation S' is a part of S; and S' is certainly a proper part of S, because there are elements in S (*e.g.* my own ego) which are different from every such thoughts *s'* and therefore are not contained in S'. Finally it is clear that if s_1 and s_2 are different elements of S, their transformations s_1' and s_2' are also different, given that the transformation is a distinct (similar) transformation. Hence, S is infinite. Which was to be proved.[11]

4.17. Once our stupor dissipates (but it is of the same order as that which seizes us in reading the first propositions of Spinoza's *Ethics*), we must proceed to a close examination of this proof of existence.

4.18. Some technical specifics. The course of the proof consists of discussing the correspondence between an "object of my thought" and the thought "this is an object of my thought" – that is to say the correspondence *between a thought and the thought of that thought, or reflection* – as a function operating between elements of the set of my possible thoughts (we could in fact identify a "possible object of my thought" with *a* possible thought). This function is "distinct" (we would now say injective), because it possesses the property (which biunivocal correspondences also possess) that two distinct elements always correspond via the function to two distinct elements. In fact, given two thoughts whose objects distinguish them from each other, the two thoughts of these thoughts are distinct (they also have distinct objects, since they think of

---

[11] [Dedekind, *op.cit.*, 66. Dedekind's text has ɸ where Badiou has *f*, and *a,b* rather than s_1, s_2 – trans.]

distinct thoughts). The result is that there is a biunivocal correspondence between thoughts in general and thoughts of the type "thought of a thought". Or, if you like, there is such a correspondence between thoughts whose object is anything whatsoever, and thoughts whose object is a thought. Now this second set forms a proper part of the set of all possible thoughts, since there are thoughts which are *not* thoughts of thoughts: the striking example Dedekind gives is that which we call 'the ego'. Thus the set of all my possible thoughts, being in biunivocal correspondence with one of its proper parts, is infinite.

4.19. Dedekind's approach *is a singular combination of DescarteS' Cogito and the idea of the idea in Spinoza.*

The starting point is the very space of the Cogito, as "closed" configuration of all possible thoughts, existential point of pure thought. It is claimed (but only the Cogito assures us of it) that something like the set of all my possible thoughts *exists*.

In the causal "serialism" of Spinoza, we find conjoined (regardless of the Dedekind's historical sources) the existence of a "parallelism" which allows us to identify simple ideas by way of their object (Spinoza says: through the body of which the idea is an idea), and the existence of a reflexive redoubling, which assures the existence of "complex" ideas of which the object is no longer a body, but another idea. For Spinoza, as for Dedekind, this process of reflexive redoubling must *go to infinity*. In fact, an idea of an idea (or the thought of a thought of an object) *is an idea*. So there exists an idea that is the idea of the idea of the idea of a body, etc.

All of these themes are necessary for Dedekind to be able to conclude the existence of an infinite system. There must be a circumscribed "site", representable under the sign of the One, of the set of my possible thoughts. We recognise in this site the soul, the "thinking thing", such as Descartes posits in the Cogito, existence, or essence (pure thought). It is necessary that an idea should be identifiable through its object, in such a way that two different ideas correspond to two different objects: this alone authorises the biunivocal character of the correspondence. Ultimately, it must be that the reflexive process should go to infinity, since if it did not there would exist thoughts without correspondents through the function, thoughts for which there were

no thoughts of those thoughts. This would ruin the argument, because it would no longer be established that to *every* element of the set of my possible thoughts S there corresponds an element of the set of my reflexive thoughts S'. Ultimately – above all, I would say – it must be that there is at least one thought that *is not* reflexive, that is not a thought of a thought. This alone guarantees that S', set of reflexive thoughts, is a *proper* part of S, set of my possible thoughts. This time, we recognise this fixed point of difference as the Cogito as such, called by Dedekind "my own ego". *That which does not allow itself to be thought as thought of a thought is the act of thinking itself, the "I think"*. The "I think" is non-decomposable; it is impossible to grasp it as a thought of *another* thought, since every other thought presupposes it.

It is therefore not an exaggeration to say that for Dedekind, ultimately, number exists in so far as the Cogito is a pure point of existence, underlying all reflection (specifically, there is an 'I think that I think') but itself situated outside of all reflection. The existential foundation of the infinite, and therefore of number, is that which Sartre calls the 'pre-reflexive Cogito'.

We discover in this tendency a variant of the Jacques-Alain Miller's thesis: what subtends number is the subject. The difference is that for Miller it is the 'process of engendering' of number that requires the function of the subject, whereas for Dedekind it is the existence of the infinite as its site. The Fregean programme of the conceptual deduction of zero and the Dedekindian programme of the structural deduction of the infinite lead back to the same point: the subject, whether as insistence of lack, or as pure point of existence. The Lacanian subject is assignable to the genesis of zero, the subject of Descartes to the existence of the infinite. As if two of the three great modern challenges of thinking number (zero, the infinite, the fall of the One), as soon as one assumes the third in the guise of a theory of sets, can only be resolved through a radical usage of that grand *philosophical* category of modernity: the subject.

4.20. Evidently, I could content myself with saying that, just as I am sufficiently Leibnizian to follow Frege, I am equally neither Cartesian nor Spinozist enough to follow Dedekind.

4.21. Against the Spinozism of Dedekind. Far from the idea of an infinite recurrence of the thought of a thought of a thought of a thought of a thought, etc., being able to found the existence of the site of number, *it presupposes it*. In fact, we have no experience of this type. Only the existence – and *by way of consequence* the thought – of the series of numbers, allows us to represent, and to make a numerical fiction of, a reflection which reflects itself endlessly. The very possibility of stating a 'thought' at, let us say, the fourth or fifth level of reflection of itself, obviously relies on the abstract knowledge of numbers as a condition. As to the idea of a reflection that "goes to infinity", it is obvious that within this is contained precisely what we are trying to demonstrate, namely the effect of infinity in thought, the only known medium for which is the mathematics of number.

4.22. As regards questions of *existence*, Spinoza himself made sure not to proceed like Dedekind. He did not at all seek to infer the existence of the infinite from the recurrence of ideas. It is, rather, precisely *because* he postulated an infinite substance that he was able to establish that the sequence that goes from the idea of a body to ideas of ideas of ideas, etc., is infinite. For him, and he was quite justified in this, the existence of the infinite *is an axiom*. His problem is rather 'on the other side', the side of the body (or for Dedekind, that of the object). Because if there is a rigorous parallelism between the chain of ideas and the chain of bodies, then there must be, corresponding to the idea of an idea, the 'body of a body', and we are unable to grasp what the reality of such a thing might be. Dedekind evades this problem because the site of thinking he postulates assumes the Cartesian closure: the corporeal exterior, the extensive attribute, does not intervene in it. But, in seeking to draw from the Spinozist recurrence a conclusive (and non-axiomatic) thesis on the infinite, it produces only a vicious circle.

4.23. Against the Cartesianism of Dedekind. It is essential to the proof that *every* thought can be the object of a thought. This theme is incontestably Cartesian: the "I think" supports the being of ideas in general as the 'material' of thought, and it is

clear that there is no idea that cannot be a thinkable idea, that is to say (since we are speaking of the set of my possible thoughts) virtually actualisable as object of my thought. But obviously this excludes the possibility that "it" could be thought without my thinking that I think that thought, and without it being even *possible* that I so think. Dedekind is Cartesian in his exclusion of the unconscious, which, since Freud, we know to think, and to think in such a way that some of its thoughts are definable precisely as those which I cannot think. 'Unconscious thoughts' are precisely those unable, at least directly, to become objects of my thought.

More generally it is doubtful, for a contemporary philosopher, whether *true* thoughts, those that are included in a generic procedure of truth, would be amenable to exposure as such in the figure of their reflection. This would be to imagine that their translation onto the figure of knowledge (which is the figure of reflection) is coextensive with them. Now the most solid idea of contemporary philosophy is precisely not to understand the process of truth except as *a gap* in knowledge. If 'thought' means: instance of the subject in a truth-procedure, then there is not a thought of this thought, because it contains no knowledge. Dedekind's approach founders on the unconscious, and does not hold firmly enough to the distinction between knowledge and truth.

4.24. Descartes himself is more prudent than Dedekind. He makes sure not to infer the infinite from reflection, or from the Cogito in itself. He does not consider, in proving the existence of God, the *totality* of my possible thoughts, as Dedekind does. On the contrary he singularises *an* idea, the idea of God, in such a way that one can contrast its local argument to the global, or set-theoretical, argument of Dedekind. The problem of Descartes is elsewhere, it is of a Fregean nature: how to pass from concept to existence? For this, an argument of disproportion *between the idea and its site* is necessary: the idea of the infinite is without common measure with its site, which is my soul – or, for Dedekind, the set of my possible thoughts; because this site, grasped in its substantial being, is finite. The singular idea of the infinite must then "come from elsewhere"; it must come from a real infinity.

We can see how ultimately the positions of Descartes and Dedekind are reversed. For Dedekind, it is the site that is infinite, because of its having to support

reflection (the capacity of the Cogito) in its going to infinity. For Descartes, it is the exterior of the site (God) that is infinite, since the site of my thought, guaranteed in its being by the Cogito, is finite, and is therefore not *capable* alone of supporting the idea of the infinite. But, in wanting to break with the finitude of the site, Dedekind forgets that this site could well be nothing but a piece of scenery, fabricated from an Other site, or that thought could well find its principle only in a *presupposition* of infinite number, of which it would be the finite and irreflexive moment.

4.25. Immanent, or argumentative critique. Dedekind's starting point is "the realm of all possible objects of my thought", which he immediately decides to call system S. But *is this domain amenable to being considered as a system, that is to say a set*? Do the "possible objects of my thought" form *a* set, a consistent multiplicity, which can be counted as one (leaving aside the thorny question of knowing *what* carries out this accounting of my thoughts)? Isn't it rather an inconsistent multiplicity, insofar as its total recollection is, for thought itself, precisely impossible? If one admits the Lacanian identification of the impossible and the real, wouldn't the 'system' of *all* possible objects of my thoughts be the real of thought, in the guise of the impossibility of its counting-for-one? Now, after having established that the "realm of all possible objects of my thought" is an infinite system, we must establish that it is a system (*a* set).

4.26. In the same way in which Russell's paradox comes to spoil Frege's derivation of number on the basis of the concept, the 'paradox' of the set of all sets – a descendant of the former – comes to break Dedekind's deduction of the existence of the infinite, and by way of consequence the deduction of the existence of N, the "simply infinite" set which is the site of number. Conceptually set out by Dedekind with impeccable inferences, the site of number does not stand the test of consistency, which is also that of existence.

4.27. Let us reason 'à la Dedekind'. Any system whatsoever (a set), grasped in abstraction from the singularity of its objects or, as Dedekind says, thought uniquely

according to "that which distinguishes" these objects (thus, their simple belonging to a system and its laws), is obviously a possible object of my thought. In consequence, within the supposed system S of all possible objects of my thought must figure, as a subsystem (subset), the system of all systems, the set of all sets. By virtue of this fact, this system of all systems is itself a possible object of my thought. Let us say to simplify that the system of all systems is a thought.

Now, this situation is impossible. In fact, a fundamental principle of Dedekind's demonstration has it that every thought gives rise to a thought of this thought, which is different from the original thought. So if there exists a thought of the set of all sets, there must exist a thought of this thought, *which is in S*, the set of all my possible thoughts. S is then *larger* than the set of all sets, since it contains at least one element (the thought of the set of all sets) that does not figure in the set of all sets. Which cannot be, since S *is* a set, and therefore must figure as an element *in* the set of all sets.

Or, once again: considered as a set or system, S, domain of all the possible objects of my thought, is an element of the set of all sets. Considered in its serial or reflexive law, S overflows the set of all sets, since it contains the thought of that thought which is the set of all sets. S is thus at once inside (or "smaller than") and outside (or "larger than") one of its elements: the thought of the set of all sets. We must conclude then, save for logical inconsistency, either that the set of all sets, the system of all systems, *is not a possible object of my thought*, as we have come to think; or, and this is more reasonable, that the domain of all possible objects of my thought is not a system, or a set. But in that case, it cannot be used to support a proof of the existence of an infinite system.

4.28. Reasoning now in a more mathematical fashion: Suppose that the set of all sets exists (which implies necessarily the existence *as set* of the domain of all possible objects of my thought). Then, since it is a set, we can separate (Zermelo's axiom, *cf.* **2.12.**) as an existent set all of the elements that have a certain property in common. Take the property 'not being an element of itself'. By means of separation this time, and therefore with the guarantee of existence already in place, we 'cut out' from the set of all sets, which we suppose to exist, the set of all the sets which are not members

of themselves. This set then exists, which Russell's paradox tells us is impossible (admitting the existence of the set of all sets which are not members of themselves leads directly to a formal contradiction, *cf*. **2.11.**). So it is impossible that the set of all sets should exist, and *a fortiori* that the domain of all my possible thoughts could be a set.

4.29. Dedekind's attempt ultimately founders at the same point as did Frege's: in the transition from the concept to the assertion of existence. And at the root of the affair is the same thing: Frege and Dedekind both seek to deduce from 'pure logic', or thought as such, not just the operative rules of number, but the fact of its existence for thought. Now, just like the empty set, or zero, *the infinite does not allow itself to be deduced*: we have to *decide* its existence axiomatically, which comes down to admitting that one takes this existence, not for a construction of thought, but for a fact of Being.

The site of number, whether we approach it, like Frege, "from below", on the side of pure lack, or like Dedekind "from above", from the side of infinity, cannot be established by way of logic, by the pressure of thought alone upon itself. There has to be a pure and simple *acknowledgement* of its existence: the axiom of the empty set founds zero, and from there, as a result, the finite cardinals exist. The axiom of infinity founds the existence of the infinite ordinals, and from there we can return to the existence of finite ordinals. The challenge posed to the moderns by the thinking of number cannot be met by a *deduction*, but only by a *decision*. And the support for this decision, as to its veridicality, does not arise from intuition or from proof. It arises from its conformity to that which being *qua* being prescribes to us. From the fact that the One is not, it follows, with regard to zero and the infinite, that there is nothing to say other than: they are.

4.30. Nevertheless, we must give Dedekind immense credit for three essential ideas.

The first is that the best approach to number is a general theory of the pure multiple, and therefore a theory of sets. This approach, an ontological one, entirely distinguishes him from the conceptual or logicist approach, such as we find in Frege.

The second is that, within this framework, we must proceed in "ordinal" fashion, erecting in thought a sort of universal series where number *will come to be grasped*. Certainly, the theory of ordinals must be removed from its overdependence on the idea of order, still very much present in Dedekind. Because, as I objected to Jacques-Alain Miller, we do not have to presume that the being of number will be awaiting us along the ordered route that we propose to it. The concept of the ordinal must be still further ontologised, rendered less operative, less *purely* serial.

The third great inspired idea of Dedekind is that to construct a modern thinking of number, a non-Greek thinking, we must *begin* with the infinite. The fact that it is vain to try and give to this beginning the form of a proof of existence is ultimately a secondary matter, compared to the idea of the beginning itself. It is truly paradigmatic to have understood that in order to think finite number, the whole natural numbers, it is necessary *first* to think, and to bring into existence – by way of a decision that follows the historial nature of being insofar as our epoch is that of the secularisation of the infinite (of which its numericisation is the first example) – infinite number.

On these three points, Dedekind is truly the closest companion, and in certain ways the ancestor, of the father – still unrecognised – of the great laws of our thought: Cantor.

5. Peano

5.1. Peano's work is not necessarily comparable, in profundity or in novelty, either to Frege's or to Dedekind's. His success lies more in the clarification of a symbolism, in the solidly assured connection between logic and mathematics, and in a real talent for discerning and denoting the pertinent axioms. One cannot speak of number without tackling at their source the famous "Peano axioms," which have become the scholarly reference for any kind of formalised introduction of the whole natural numbers.

5.2. Even though, from in the opening of his *Principles of Arithmetic*[1], – written, deliciously, in Latin – Peano speaks of "questions that pertain to the foundations of mathematics", which he says have not received a "satisfactory solution", the approach he adopts is less that of a fundamental meditation than of a "technicisation" of procedures, with a view to establishing a sort of manipulatory consensus (something in which, in fact, he perfectly succeeds). This is the sense in which we ought to understand the phrase: "The difficulty has its main source in the ambiguity of language." To expose number in the clarity of a language – an artificial clarity, certainly, but legible and indubitable – this is what is at stake in Peano's work.

5.3. Substantially, the approach is modelled on Dedekind's. We 'start' from an initial term, which, as with Dedekind, is not zero but one. We put 'to work' the successor function (which is denoted in Peano according to the additive intuition: the successor

---

[1] The reference text for Peano is a text published in latin in 1889, and of which the title in French is: "Les Principes de l'arithmetique". The English translation of this text is found in J.Van Heijenhoort(ed), *From Frege to Godel*, *op cit*, p83-97.

of *n* is written *n*+1). We rely heavily on induction, or reasoning by recurrence. But whereas Dedekind, who works in a set-theoretical framework, *deduces* the validity of this procedure, in Peano it is treated purely and simply as an axiom. We decide that:

- If 1 possesses a property,

- And if it is true that, when *n* possesses a property, then *n*+1 also possesses it,

- Then, *all* numbers *n* possess the property.

Armed with this inductive principle and with purely logical axioms whose presentation he has clarified, Peano can *define* all the classical structures of the domain of whole numbers: total order and algebraic operations (addition, multiplication).

5.4. The axiom of induction, or of recurrence, marks the difference in thinking between Peano and Dedekind on the crucial issue of the infinite. Treated as a simple operative principle, recurrence actually permits legislation over an infinite totality *with no mention of its infinity*.

In fact it is clear that there is infinity of whole numbers. To speak of "all" these numbers therefore means to speak of an actual infinity. But in Peano's axiomatic apparatus, this infinity is not introduced as such. The axiom of recurrence permits us, from a *verification* (1 possesses the property) and an implicative *proof* (*if n* possesses the property, *then n*+1 also possesses it), to conclude that "all numbers possess the property", without having to inquire as to the extension of this "all". The universal quantifier here masks the thought of an actual infinity: the infinite remains a latent form, inscribed by the quantifier without being released into thought.

Thus Peano introduces the concept of number without transgressing the old prohibition on actual infinity, a prohibition that still hangs over our thought even as it is summoned to its abolition by the modern injunction of being. Peano's axiomatic *evades* the infinite, or the explicit mention of the infinite.

For Dedekind, on the other hand, not only the concept of the infinite, but also its existence, is absolutely crucial. Dedekind says this explicitly in a letter to Keferstein:

> After the essential nature of the simply infinite system, whose abstract type is the number sequence N, had been recognized in my analysis…the question arose: does such a system *exist* at all in the domain our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs.[2]

5.5. Peano does not enter into questions of existence. As soon as a system of axioms gives its rule to operative arrangements, we are at liberty to ask about the *coherence* of that system; we need not speculate on the being of that which is thus interrogated. The vocabulary of the "thing", or object, common to Frege and Dedekind (even if it is a matter of "mental things" in the sense of Husserl's noematic correlate) is dropped in Peano's work in favour of a somewhat "postmodern" disposition where the sign reigns. He writes, for example: "I have denoted by signs all ideas that occur in the principles of arithmetic, so that every proposition is stated only by means of these signs." If the latent model of Dedekind and of Frege is philosophical (a "philosophy as rigorous science"), Peano's is directly algebraic: "With these notations, every

---

[2] This passage is taken from a letter from Dedekind to Keferstein, dating from 1890. The English translation is in *op cit*, p.98 [quote from p.101 – trans.]

proposition assumes the form and the precision that equations have in algebra [...] the procedures are silimar to those used in solving equations."[3]

Peano proposes an 'economy of number' which is an economy of signs, whose paradigm is algebraic, whose transparency is consensual, and whose operative effectiveness is therefore not in doubt. He thus participates forcefully in that movement of thought, victorious today, which wrests mathematics from its antique philosophical pedestal and represents it to us as a grammar of signs where all that matters is that the code should be made explicit. Peano prepares the way from afar – by eliminating all ideas of a being of number, and even more so those of number *as* being – for the major theses of Carnap which reduce mathematics, treated as a 'formal language' (as opposed to empirical languages), to being, not a science (because according to this conception every science must have an 'object'), but the syntax of the sciences. Peano is inscribed in the general movement of thought of our century – forged, in fact, at the end of the 19th century – whose characteristic gesture is the destitution of Platonism in that which had always been its bastion: mathematics, and especially the Idea of number.

5.6. We see here, as if in the pangs of its birth, the real origin of that which Lyotard calls the "linguistic turn" in western philosophy, and which I call the reign of the great modern sophistry: if it is true that mathematics, the highest expression of pure thought, in the final analysis consists of nothing but syntactical apparatuses, grammars of signs, then *a fortiori* all thought is under the constitutive rule of language.

---

[3] [*ibid.*, p85 – trans.]

It is certain that, for Plato, the subordination of language to 'things themselves', which he deals with for example in the *Cratylus*, has as its horizon of certitude the ontological vocation of the matheme. There is no upholding the pure empire of the sign if number, which we indicate with just a simple stroke, is, as Plato thought, a form of Being. Inversely, if number is nothing but a grammar of special signs, ruled by axioms without foundation in thought, then it is probable that philosophy must first and foremost (as Deleuze diagnoses it in Nietzsche) be a thinking of the force of signs. Either truth, or the arbitrariness of the sign and the diversity of syntactical games: this is the central choice for contemporary philosophy. Number occupies a strategic position in this conflict, because it is simultaneously the most generalised basis of thought, and that which demands most abruptly the question of its being.

Peano's axiomatic, impoverished in thought but strong in effect, a grammar which subdues number, the organising principle of an operative consensus, a skilful mediation of the infinite in the finitude of signs, is something of a beneficent artefact for modern sophistry.

5.7. Every purely axiomatic procedure introduces *undefined* signs, of which there can be no other presentation in thought apart from the codification of their *usage* by axioms. Peano is hardly economical with these "primitive" signs: there are four, in fact (I remind you that set theory has recourse to one single primitive sign, membership, $\in$, which denotes presentation as such):

> Among the signs of arithmetic, those that can be expressed by other signs of arithmetic together with the signs of logic represent the ideas that we can define. Thus, I have defined all signs except for four [...] If, as I think, these four cannot

be reduced any further, it is not possible to define the ideas expressed by them through ideas assumed to be known previously.[4]

These four irreducible signs[5] are:

1) The sign N, which "means *number (positive integer)*".

2) The sign 1, which "means *unity*".

3) The sign *a+1*, which "means *the successor of a*".

4) The sign =, which "means *is equal to*".

Peano thus explicitly renounces all definition of number, of succession, and of 1. (One might treat separately the case of the sign = : it is in point of fact a question of a logical sign, not of an arithmetical one. Peano himself writes: "We consider this sign as new, although it has the form of a sign of logic"[6]). This is obviously the ransom to be paid for operative transparency. Where Frege musters all thought to the attempt to *understand* the revolutionary statement "zero is a number", Peano simply notes (it is the first axiom of his system): $1 \in N$, a formal correlation between two undefined signs that "means" (but according to what doctrine of signification?) that 1 is a number. Where Dedekind generates the site of number as space of usage or really existing infinite chain, and of the biunivocal function, Peano notes[7]: $a \in N \rightarrow a+1 \in N$, an implication that involves three undefined signs, and which "means" that, if *a* is a number, its successor is also a number. The force of the letter is here at the mercy of signification. And the effect is not one of obscurity, but rather one of an excessive limpidity, a cumbersome levity of the trace.

---

[4] [*ibid*. p.85 – trans.]
[5] [*ibid*. p.94 – trans.]
[6] [*ibid*. – trans.]

5.8. In the poem, the obscure is born of that which, as a breaking open of the signifier, at the limits of language, disseminates the letter. In the pure axiomatic of Peano, the retreat of sense proceeds from the fact that the force of the letter is turned upon itself, and that there is *nothing outside*, of which it could be the thought. Peano would economise every confrontation with the latent poem whose absence number – astral figure of being ("cold with forgetfulness and desuetude, a constellation"[8]) – unfailingly instigates, and whose effect Frege and Dedekind unconsciously preserve in the desperate attempt to conjure into Presence now zero, now the infinite.

5.9. Peano's axiomatic is a shining success story of the tendency of our times to see nothing in number except for a network of operations, a manipulable logic of the sign. Number, Peano thinks, *makes signs* about the sign, or is the Sign of signs.

From this point of view, Peano is as one with the idea that the universe of science reaches its apex in the forgetting of being, homogenous with the reabsorption of numericality into the unthought of technical will. Number is truly *machinic*. This is why it can be maintained that the success of Peano's axiomatic participates in the great movement that has *given up* the matheme to modern sophistry, by unbinding it from all ontology, and by situating it within the resources of language alone.

5.10. It will be a great revenge upon this operation to discover, with Skolem, and then Robinson[9] the semantic *limits* of the grammar of signs to which Peano had reduced

---

[7] [*ibid*. (Axiom 6) – trans.]
[8] [Mallarmé, *ibid*, p233 – trans.]
[9] Regarding these question one can read chapter X (purely historical) of A.Robinson, *Non-standard analysis* (North-Holland Publishing Company, revised edition, 1974). Robinson recognises that "the work of Skolem on non-standard models of arithmetic has become the most important factor in the creation of non-standard analysis" (p.278).

the concept of number. We know today that such an axiomatic admits of "non-standard" models, whose proper being is very different from all that we intuitively understand by the idea of whole natural number. So that Peano's system admits of models where there exist "infinitely large" numbers, or models of which the type of infinity exceeds the denumerable. Peano arithmetic is susceptible to "pathological" interpretations; it is powerless to establish a univocal thought in the machinism of signs. Every attempt to reduce the matheme to the sole spatialised evidence of a syntax of signs runs aground on the obscure prodigality of being in the forms of the multiple.

5.11. The essence of number does not allow itself to be spoken, either as the simple force of counting and of its rules, or as the sovereignty of graphisms. We must pass into it through a meditation on its being.

N is not an "undefined" predicate, but the infinite site of exercise of that which succeeds the void (or zero), the existential seal which strikes *there* where it insists on succeeding.

That which "begins" is not the 1 as opaque sign of "unity", but zero as suture of all language to the being of the situation of which it is the language.

Succession is not the additive coding of a +1, but a singular disposition of *certain* numbers, which *are* successors, rather that not being so, and which are marked in their being by this disposition. We must also know that zero and the infinite are precisely *that which does not succeed*, and that they are this in their being, differently, although both may be, by virtue of this fact, on the edges of a Nothingness.

For a philosophical commentary on these developments, cf. A Badiou, "Infinitesimal Subversion", in *Cahiers pour l'analyse*, no 9, Paris, ed. de Seuil, Summer 1968.

Number is neither that which counts, nor that with which we count. This regime of numericality organises the forgetting of number. To *think* number requires a overturning: it is because it is an unfathomable form of being that number prescribes to us that feeble form of its approximation that is counting. Peano presents the inscription of number, which is *our* infirmity, our finitude, as the condition of its being. But there are more things, infinitely more, in the kingdom of Number, than are dreamt of in Peano's arithmetic.

6. Cantor: The "Well-Ordered" and the Ordinals

6.1. The ordinals represent the general ontological horizon of numericality. Following the elucidation of the concept of the ordinal, with which we will presently occupy ourselves, this principle governs everything that we shall say, and it is well said that in this sense, Cantor is the veritable founder of the contemporary thinking of number. Actually, Cantor[1] considered that the theory of ordinals constituted the heart of his discovery. Today, the *working mathematician*, for whom it suffices *that there are* sets and numbers and who does not worry at all about what they *are*, thinks of the ordinals rather as something of a curiosity. We must see in this mild disdain one of the forms of submission of the mathematician, insofar as he or she is exclusively *working*, to the imperatives of social numericality. Specialists in mathematical logic or set theory are doubtless an exception, even if they themselves often regret this exception: in spite of themselves, they are the closest to the injunction of Being, and for them the ordinals are essential.

6.2. I have said, in connection with Dedekind, that in the philosophical discourse that falls to us, we must assume an "ontologisation" of the ordinals as complete as possible. In fact, the presentation of this concept by Dedekind or Cantor relates it essentially to the notion of well-orderedness – still very close to a simple serial or operative intuition of number.

6.3. Every schoolboy knows that given two different whole numbers, one of them is larger and the other smaller. And he knows also that when one proposes a 'bunch' of numbers, there is one and one only that is the smallest of the bunch.

---

[1] The clearest articulation by Cantor of his ordinal conception of numbers is found in a letter to Dedekind in 1899. *Cf*. The English translation of the key passages of this letter in the collection edited by van Heijenhoort, *From Frege to Gödel, op.cit* p113-117. Cantor demonstrates an exceptional lucidity as to the philosophically essential distinction between consistent multiplicities and inconsistent multiplicities. It is to him, in fact, that we owe this terminology.

This serial knowledge, if one abstracts its general properties, gives rise to the concept of the well-ordered set.

6.4. A "well-ordered" set is a set for which:

1) Between the elements of the set, there is a relation of *total order*; if we have two elements, *e* and *e'* and if < denotes the relation of order, then either *e< e'*, *e'<e*, or *e= e'*; no two elements are 'non-comparable' by this relation.

2) Given any one non-empty part of the set so ordered, there exists a *smallest element* of this part (an element of this part that is smaller than all the others). If P is the part considered, there exists *p*, which belongs to P, and which is such that, for every other *p'* belonging to P, we have *p < p'*. This element *p* will be called the minimal element of P.

If an element *p* is minimal for a part P, it *alone* possesses that property. Because if there were another, a *p'* different from *p*, then because the order is total we would have to have either *p<p'*, and so *p'* would not be minimal; or *p'<p*, and *p* would not be. One can thus without hesitation speak of the 'minimal element' of a part P of a well-ordered set.

We can see then that the general concept of the well-ordered set is but a sort of extrapolation from that which the schoolboy observes in the most familiar numbers: the whole natural numbers.

6.5. A good image of a well-ordered set is the following. Take E, such a set. You 'begin' with the smallest element of E, which exists given condition 2. Call this element 1. You consider the part of E obtained by removing 1, the part (E−1). It too has a minimal element, which comes in a certain sense *straight after 1*. Call this element 2. Consider the part of E obtained by removing 1 and 2, the part (E−(1,2)). It has a minimal element, call it 3, etc. A well-ordered set presents itself like a chain, so that every link of the chain follows ("follows" means to say: comes just after in the

relation of total order) only one other, well determined (it is the minimal element *of that which remains*).

6.6. Cantor's stroke of genius[2] was to not limit this image to the finite, and thus to introduce infinite numerations. He had the following idea: If I suppose the existence – *beyond* the series 1,2,3, ..., *n*, *n*+1,..., of whole numbers which is the "first" well-ordered set, the matrix of all the others – of an "ordinal infinite number" ω, and declare it larger than all the numbers that precede it, then what prevents me from *continuing*? I can very well treat ω as the minimal element of a well-ordered set that comes in some sense *after* the set of whole numbers. And I can consider the "numbers" ω+1, ω+2, ..., ω+*n*, .., etc. I will arrive eventually at ω+ω, and will continue once again. No stopping-point is prescribed to me, even if I am dealing with a sort of total series, of which each term is the possible measure of every existent series. This term indicates to me in fact that *however many it has before it*, it *numbers every series of the same length*.

6.7. Let us allow ourselves to call *ordinal* the measure of length of a well-ordered set, from its minimal element to its "end". The "entire" series of ordinals would then provide us with a scale of measurement for these lengths. Each ordinal would represent a possible structure of well-orderedness, determined by the way in which the elements succeed, and by the total number of these elements. This is why we say that an ordinal, whether finite (the ordinals which come before ω, and which are quite simply the whole natural numbers) or infinite (those ordinals which come after ω), number a "type of well-orderedness".

---

[2] On this point, one naturally should refer to the work of Alexander Koyré.

6.8. To give a technical grounding for this idea, we consider the class of well-ordered sets that are isomorphic to one of the sets among them (and therefore isomorphic to each other). What are we to understand from this?

Take two well-ordered sets, E and E', < the relation of order of E, and <' the relation of order of E'. I will say that E and E' are isomorphic if there exists a biunivocal correspondence f (*cf.* **4.5.**) between E and E', such that, when $e_1 < e_2$, in E, then $f(e_1) <' f(e_2)$ in E'.

We can see that f projects the order of E into the order of E', and, what is more, since f is biunivocal, there are "as many" elements in E' as in E. We can therefore say that E and E', considered strictly from the point of view of their well-orderedness, and abstracted from the singularity of their elements, are identical: the "morphism" (form) of their well-orderedness is "iso" (the same), as the correspondence f assures us.

Each class of well-ordered sets isomorphic to each other represent in fact *a* well-orderedness, the one that is common to all the sets of that class. It is *this* well-orderedness that can be represented by an ordinal.

Thus an ordinal is the mark of a possible figure (of a form, of a morphism) of well-orderedness, isomorphic to all the sets that take that form. An ordinal is *the number or the cipher of a well-orderedness.*

6.9. This conception, already moving strongly in the direction of a determination of a *horizon* of being of all number, in the form of a universal scale of measurement for forms of well-orderedness, nevertheless presents some serious difficulties; the first among them technical, the remainder philosophical.

6.10. The technical difficulties are three in number, three questions to which we must respond:

1) Which is the *first* term of the total series of ordinals, the initial link that "anchors" the whole chain? This is the conceptual question of zero or the empty set, alone able to number the series of no length, the series without element, the well-orderedness that orders *nothing*. This is the question that caught out Frege.

2) What exactly is the procedure of thought that allows us to suppose a beyond of the series of finite whole natural numbers? What is the gesture by which we *pass out of* the finite, and declare ω, the first ordinal which will not be a whole natural number, the first mark of a well-orderedness that describes the structure of a non-finite set? This is the existential question of the infinite, upon which Dedekind foundered.

3) Does the universal series of ordinals – the scale of measurement of all length, whether finite or infinite, the totality of specifications of well-orderedness – *exist* in the set-theoretical framework? Isn't it – like the "system of all the possible objects of my thought" introduced by Dedekind – an *inconsistent* totality, one that thought *cannot* take as one of its possible objects? This is the question of counting-for-one an "absolute" totality. It is thus the problem of the desolation of the One as soon as one claims to "count" the universe of discourse.

And here we are returned to the three challenges of the modern thinking of number: zero, the infinite, and the non-being of the One.

6.11. The third problem is rapidly revealed to be without positive solution. One can in fact *prove* (something that was at one time stated as a "paradox", that of Burali-Forti) that the ordinals *do not form a set*, that they do not allow themselves to be collected in a multiple that can be counted-for-one. The idea of "all" the ordinals is inconsistent, impossible; it is to this extent the real of the horizon of the being of number.

This proof is very much related to that which refutes the manner in which Dedekind tries to prove the existence of an infinite set (*cf*. **4.28.**): the set of "all" the ordinals must itself be an ordinal, and thus it would be *inside* itself (since it is a set of

all the ordinals) and *outside* itself (since it is not counted in the series which it totalises). We would therefore be prohibited from speaking of a "set of ordinals" without qualification. Which is precisely to say: "being an ordinal" is a property *which has no extension.* One can *verify* that a certain object is an ordinal (possesses the property), but one cannot *count for one* all the objects that have this property.

6.12. I have said enough, in my critique of Frege and Dedekind, for one to imagine the treatment of problems 1 and 2: the existence of zero, or the empty set, and the existence of an infinite set can in no way be deduced from "purely logical" presuppositions. They are axiomatic decisions, taken under the constraints of the historial injunction of being. The world of modern thought is nothing other than the effect of this injunction. Beginning in the Renaissance, by way of a rupture with the Greek cosmos[3], it became *necessary* to think anything whatever in accordance with our assumptions of ontological exigency, to assume:

- That the proper mode under which every "given" situation is sutured to is being is not Presence, the foreclosure of that which is pro-posed within its limit, but pure subtraction, the unqualifiable void; in that form of being which is number, this is to say: "zero exists", or, in a style more homogenous with the ontological creation of Cantor: "there exists a set which has no elements";

- That, in their quasi-totality, and by way of rupture with the mediaeval tradition which reserves this attribute for God alone, given-situations are infinite; in such a way that, far from being a predicate whose force is that of the sacred, the infinite is a *banal* determination of being, such as it is proffered as pure multiplicity under the law of a count-for-one. In the form of being which is number, this is said: "an infinite set exists"; or more technically: "an ordinal exists which is not a whole natural number". Or in other words, "ω exists".

---

[3] On this point, one naturally should refer to the work of Alexander Koyré.

6.13. It had to wait practically until the beginning of the twentieth century before these decisions relating to zero and to the infinite would be recognised in themselves (under the names of the axiom of the empty set and the axiom of infinity), although they had operated in thought for three hundred years. But this is not surprising: We can observe a veritable *philosophical* desperation constantly putting these imperatives into reverse, whether in the intellectual dereliction of the theme of finitude, or in the nostalgia for the Greek ground of presence. It is true that, when we are dealing with pure declarations, decided in themselves, these declarations exhibit the fragility of their historicity. No *argument* can support them. What is more, certain truth procedures, in particular the political, art, and love, are no longer equal to such axioms, and are therefore sidelined, *remaining Greek.* They cling to Presence (art and love), refusing without fail the statement "zero is the proper numeric name of being" in order to give tribute to the obsolete rights of the One. Or (the political) they manage finitude, corroding day by day the statement "the situation is infinite", in order to valorise the corrupted authority of necessities.

6.14. Concerning number as they do, the two axioms of the void and of the infinite architect all thought in terms of number. The pure void is that which ensures *that there is* number, and the infinite that by which it is affirmed that it number is the measure of the thought of *every* situation. That it is a matter of axioms and not of theorems signifies that the existence of zero and of the infinite is that which being prescribes to thought in order that the former might exist *in the ontological epoch* of such an existence.

In this sense, the current force of reactive, archaic and religious will are marked necessarily by an irremediable opacity of number – which, not ceasing to rule over us, since this is the epochal law of being, nevertheless becomes unthinkable for us. Number may exist as form of being but, as a result of the void and the infinite being totally secularised, thought can no longer exist in the form and with the force that the epoch prescribes to it. So number manifests itself, without limit, as a tyranny.

6.15. The principal philosophical difficulty of the Cantorian concept of the ordinals is as follows: In the presentations which bind it to the concept of well-orderedness, the theory of ordinals rather seems to "generalise" the intuition of whole natural number that allows us to *think* the being of number. It draws its authority from that which it claims to elucidate. The idea of well-orderedness is in effect less a foundation of the concept of number than it is deduced from the lacunary and finite experience of numerical immediacy, which I incarnated (in **6.3.**) in the sympathetic figure of the schoolboy.

If we truly wish to establish the being of number as the form of the pure multiple, to 'deschoolboy' it (which means also to subtract the concept from its ambient numericality), we must distance ourselves from operative or serial manipulations. These manipulations, so tangible in Peano, project onto the screen of modern infinity the quasi-sensible image of our domestic numbers, the 1, followed by 2, which precedes 3, and then the rest. The establishing of the correct distance between thought and countable manipulations is precisely what I call the ontologisation of the concept of number. From the point where we presently find ourselves, it takes on the form of a most precise task: the ontologisation of the 'universal' series of the ordinals. To proceed with it, we must abandon the idea of well-orderedness and think ordination, ordinality, in an intrinsic fashion.

It is not as a measure of order, or of disorder, that the concept of number presents itself to thought. We need an immanent determination of its being. The question is formulated for us: which predicate of the pure multiple, graspable outside of all serial engenderment, founds numericality? We do not want to count; we want to think counting.

2. Concepts: Natural Multiplicities

7. Transitive Multiplicities

7.1. What will allow us to abandon every *primitive* tie between number and order or seriality is the concept of the transitive set. Only this structural operator, of an essentially ontological nature, enables an intrinsic determination of number as a figure of natural being. Thanks to it, we are escape from the loop of the deduction of the concept (Frege), of the subject as causality of lack in serial engenderment (Miller), of the existence of the infinite (Dedekind), or of the 'schoolboy' intuition of well-orderedness (Cantor).

7.2. However mysterious this concept is at first glance, in any case its being unrelated to any intuitive idea of number is to my eyes a virtue. It proves that in it we grasp something that breaks the circle of an ontological elucidation of number entirely transparent in its pure and simple presupposition. We have seen that this circle reoccurs in Frege as in Dedekind, and that the Cantorian conception of the ordinals as types of well-orderedness still remains under its influence. We shall see moreover that the legitimacy of the concept of transitivity *for philosophical thought* leaves no doubt.

7.3. To understand what a transitive set is, it is essential effectively to penetrate the distinction – of which it would not be an exaggeration to say that it supports all post-Cantorian mathematics – between *membership* of an element of a set and *inclusion* of a part. This distinction is rudimentary, but it implies such profound consequences that it remained obscure for a long time.

7.4. A set is 'made out of elements', is the 'collection' (in my language, the count-for-one) of its elements.

Given the set E, and e one of the elements of which it 'makes' a set, this is denoted $e \in E$, and we say that e is a member of E, that $\in$ is the sign of membership.

If you now take "together" *many* elements of E, they form a part of E. Given E' the set of these elements, E' is a part of E, this is denoted $E' \subset E$, and one says that

E' is included in E, that ⊂ is the sign of inclusion.

Every element of a part E' of E is an element of E. In fact this is the definition of a part: E' is *included* in E when all the elements that are *members* of E' are *also members* of E. So we see that inclusion is defined in terms of membership, which is the only 'primitive' sign of set theory.

The classic (misleading) image is drawn like this:
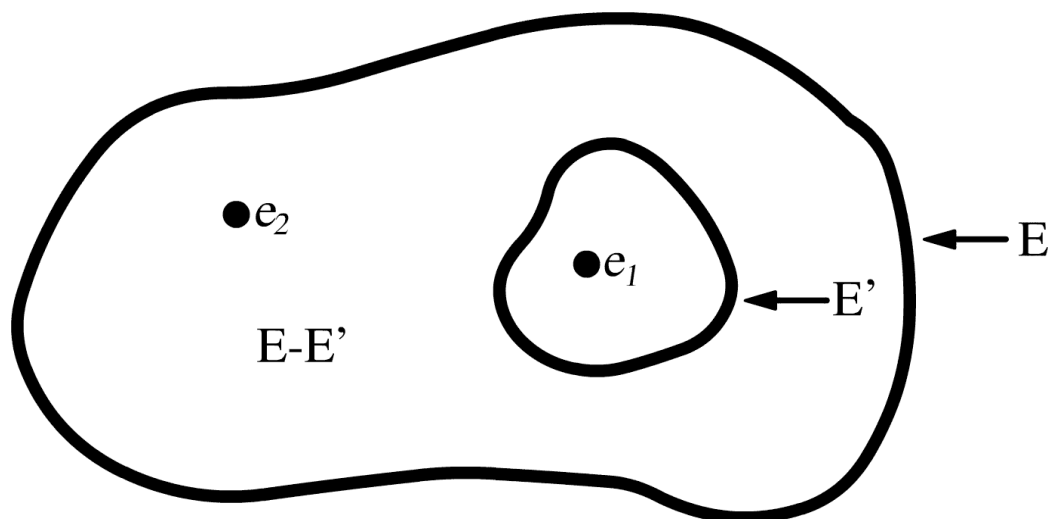
We read that E' is a part of E, that e_1 is at once (as is every element of E') an element of E' and an element of E, and that e_2 is an element of "the whole" E, but not of the part E'. We also say that e_2 is a member of the *difference* of E and E' which is denoted E – E'.

7.5. Is it possible that an element that *is a member* of the set E, could also be a part of this set, could also be *included*? This seems totally bizarre, above all if one refers to the image above. But this sentiment misses the most important point: which is that an element of a set can obviously be (and is always, even) itself a set. Consequently, if *e* is a member of E, and *e* is a set, the question occurs whether an element of *e* is, or is not, in its turn an element of E. If all the elements of *e* are also elements of E, then *e*, which is an element of E, is also a part of E. It belongs to E *and* is included in E.

7.6. Suppose for example that V is the set of living beings. My cat is a member of this set. But a cat is composed of cells, which one might maintain are all living beings. So my cat is at once *a* living being and *a set* of living beings. He is a member of V (*qua* one, *this* living cat), and he is a part of V – he is included in V – (*qua* group of living cells).

7.7. Forget cats. Consider the three "objects" as follows:

– the object e_1;

– the object e_2;

– the object which is the "putting together" of the first two, and which we denote by (e_1,e_2). We say that this is the *pair* of e_1 and e_2.

Form a set from these three objects. In the same way, we denote it: $(e_1,e_2,(e_1,e_2))$. We say that this is the *triplet* of e_1 and e_2 and the pair (e_1,e_2). We denote it T. Note that the three elements that are *members* of this triplet are e_1, e_2, and (e_1,e_2).

Since e_1 and e_2 are members of T, if I "put them together", I obtain a *part* of T. Thus, the pair (e_1,e_2) which is the "putting together" of these two elements of T, is *included* in T. But, moreover, we can see that it is an element of it, that it is a *member*. Thus we have constructed a very simple case of a set of which an element is also a part. In the set T, the pair (e_1,e_2) is simultaneously in a position of membership and of inclusion.

7.8. We know, by a famous theorem of Cantor's, that there are *more* parts than elements in any set E whatsoever. This is what I call the excess of inclusion over membership, a law of being *qua* being whose consequences for thought are immense, because it affects the fundamental categories that inform the couplets One/Multiple and Whole/Part. It is therefore impossible that every part should be an element, that everything that is included should also be a member: there are *always* parts that are not elements.

But we can pose the question from the other direction: since we can see that it is possible in certain cases (for example my cat for the set V of living beings, or the pair (e_1, e_2) for our triplet T) for an element to be a part, is it possible that *all* elements could be parts, that everything that is a member could be included? This is not the case for T: the element e_1 taken alone, for example, is not a part of T.

Can we produce a non-empirical example (because my V, my cat and its cells are rationally suspect) of a set of which all the elements would be parts?

7.9. Let us go back a little, to the empty set. We have proposed (in **2.18.**) the axiom 'there exists a set which has no elements', that is a set to which nothing belongs. We are going to give to this set, 'empty' foundation-stone of the whole edifice of multiple-being, a proper name, the name "0".

The following statement is of an extremely subtle nature: *the empty set is a part of every set*; 0 is included in E whatever E might be. Why? Because, if a set F *is not* a part of E, it is because there are elements of F which are not elements of E (if every element of F is an element of E, by definition F is a part of E). Now 0 has no elements. So, it is impossible for it *not to be* a part of E. The empty set is 'universally' included, because nothing in it can prevent, or deny, such inclusion.

To state it differently: to demonstrate that F is not a part of E requires the differentiation, *within* F, of at least one element: that element which, not being an element of E, proves that F cannot be included "entirely" within E. Now the void does not tolerate any differentiation of this sort. It is in-different, and, because of this, it is included in every multiplicity.

7.10. Consider the two following "objects":

- the empty set, 0;

- the set of which the one and only element is the empty set, which we call the *singleton* of the empty set, and denote (0).

Note well that this second object is *different* from the empty set itself. In fact, the empty set has *no* elements, whereas the singleton has one, which is precisely the empty set. The singleton of the empty set "counts for one" the empty set, so that the empty set does not count for nothing (this indicates a subtle distinction between "counts for nothing", which is 0, and "counts nothing", which is (0). Plato already played on this distinction in his *Parmenides*).

7.11. An additional remark as regards the singleton (the singleton "in general", not the specific singleton of the empty set): Take a set E, and *e* one of its elements (we have $e \in E$). I can say that the singleton of *e*, denoted (*e*), is a *part* of E, that one has $(e) \subset E$.

What is the singleton of *e*, in fact? It is the set whose unique element is *e*. If by consequence *e* is an element of E, 'all' the elements of the singleton (*e*), namely the unique element *e*, are elements of E, and thus (*e*) is included in E.

7.12. 'Put together' our two objects, the empty set denoted 0 and the singleton of the empty set, denoted (0). We obtain the pair (0,(0)), which we will denote D. This time, the *two* elements of the pair D are also parts, everything that is a member of D is also included in D. In fact, the first element, 0, the empty set, is included in any set whatsoever (*cf.* **7.9.**). Specifically, it is a part of the pair D. But, what is more, since 0 is an *element* of D, its singleton (0), is a *part* of D (*cf.* **7.11.**). But (0) is just the second element of D. Thus this element is also included in D. The set D is such that every element of D is a part; everything that is a member of D is included in D.

7.13. As Cantor's theorem enables us to predict, there are parts of D that are not elements of D. For example, the singleton of the element (0) of D is a part of D, as is every singleton of an element (*cf.* **7.11.**). We can refer to ((0)) as the "singleton of the

singleton". Now this object is *not* one of the two elements of D.

7.14. Important definition: we say that a set T is *transitive* if it is like the set D which we have just built: if all of its elements are also parts, if everything that is a member of it is also included in it, if every time we have $t \in T$ we also have $t \subset T$.

7.15. Transitive sets *exist*, without doubt. Perhaps V, the set of living beings; certainly the set $(0,(0))$, which is transparent, translucid even, constructed as it is from the void (the pair of the void and singleton of the void, the void as such and the void as one).

7.16. Modernity is defined by the fact that the One is not (Nietzsche said that "God is dead", but for him the One of Life occupies the place of his death). So, for we moderns (or "free spirits"), the Multiple-without-One is the last word on being *qua* being. Now the thought of the pure multiple, of the multiple considered in itself, without consideration of what it is the multiple of (so: without consideration of any *object* whatsoever), is called: "the mathematical theory of sets". In result, then, every major concept of this theory can be understood as a concept of modern ontology.

What is to be understood by the concept of the transitive set?

7.17. Membership is an ontological function of *presentation*, indicating *that which* is presented in the count-for-one of a multiple. Inclusion is the ontological function of *representation*, indicating multiples re-counted as parts in the framework of a representation. It is a problem of great importance (the problem of the *state of a situation*) that is determined by the relation between presentation and representation.

Now, a transitive set represents the *maximum possible* equilibrium between membership and inclusion, the element and the part, $\in$ and $\subset$. Transitivity thus expresses the most superior type of ontological stability; the strongest correlation between presentation and representation.

There is always an excess of parts over elements (Cantor's theorem), there always exist parts of a set which are not elements of that set. Thus we obtain the

maximal correspondence between membership and inclusion precisely when *every* element is a part: when the set considered is transitive.

This strong internal framework of the transitive set (the fact that everything that it presents in the multiple that it is, it represents a second time in the form of inclusion), this equilibrium, this maximal stability, leads me to say that transitive sets are "normal", taking "normal" in the double sense of non-pathological, stable, strongly equilibriated, that is to say not exposed to the disequilibrium between presentation and representation, a disequilibrium whose effective form is the evental caesura; and submitted to a norm, that of a maximally-attenuated correspondence between the two major categories of ontological immanence: membership and inclusion.

7.18. The concept of transitive multiplicity would constitute the *normal* basis of the thinking of number. Transitivity is at once that which makes of number a cut in the equilibrial fabric of being, and that which sets the conditions for this cut.

8. Von Neumann Ordinals

8.1. Let us consider more closely the set D, introduced in **7.12.**, written as (0,(0)), which is the pair of the void and the singleton of the void.

We have seen that this set D is transitive: its two elements, 0 and (0) are also parts of D. We can make a more specific remark: *these two elements are also transitive sets*.

– That (0) is transitive is self-evident: the only element of the singleton (0) is 0. Now, 0 is a "universal" part included in every set, and in particular included in the set (0). So the unique element of (0) is also a part of (0), and in consequence (0) is a transitive set.

– That 0, the empty set, is transitive results from its negative "porosity" to every property, which already makes it a part of any set whatsoever (*cf*. **7.9.**): a transitive set is one all of whose elements are also parts. Thus a set that is not transitive has at least one element that is not a part. Now 0 has no elements. So it cannot *not* be transitive. Consequently, it is.

We have constructed with our set D, not only a transitive set, but a transitive set of transitive sets: this transitive set "puts together" transitive sets. Both 0, (0), and their pair (0,(0)), are transitive.

8.2. A truly fundamental definition: *A set is an ordinal (in von Neumann's sense[1])* if it is like D, that is, if it is transitive and all of its elements are transitive.

8.3. This definition achieves the technical part of the ontologisation of the concept of the ordinal. It is not a question here of well-orderedness, of the image of the series of whole natural numbers, or of an operative value. Our concept is purely immanent. It describes a certain internal structural form of the ordinal, a form that connects

---

[1] John von Neumann gave a definition of ordinals independent from the concept of well-orderedness for the first time in a german article of 1923, entitled "On the introduction of transfinite numbers". This article is reproduced, in English translation, in J. van Heijenhoort (ed), op cit., p346-354.

together in a singular fashion the two crucial ontological operators, membership and inclusion, $\in$ and $\subset$.

The set D, which we use as an exemplary case, is therefore an ordinal. We can lift a corner of the veil on its identity: it is *the number Two*. Moreover, this Two allows us to affirm that von Neumann ordinals *exist*.

8.4. Before deploying this new concept of the ordinal, let us proceed with a preliminary examination of the status of its definition and the reasons why the ordinals constitute the absolute ontological horizon of *all* numbers.

8.5. I have indicated (**7.16.**) that a transitive set is the ontological scheme of the "normal" multiple. Taking into account the fact that the excess of representation over presentation is irremediable, transitivity represents the maximal equilibrium between the two.

Now, not only is an ordinal transitive, but all of its elements are also transitive. An ordinal disseminates to the interior of a multiple that normality that characterises it. It is a normality of normalities, an equilibrium of equilibria.

A truly remarkable property results from this, which is that *every element of an ordinal is an ordinal*.

In fact, take an ordinal[2] W and *x*, a member of that ordinal (we have $x \in W$). W being an ordinal, all of its elements are transitive, so *x* is transitive. For the same reason (the ordinality of W) W is itself transitive, so *x*, an element of W, is also a part of W and we have $x \subset W$. As a result, all the elements of *x* are elements of W. And, just as all the elements of W are transitive, the same follows for all the elements of *x*. The set *x* is thus a transitive set of which all the elements are transitive: it is an

---

The definition of ordinals on the basis of transitive sets seems to have been taken up again in an article in English published in 1937 by raphael M.Robinson, entitled "The theory of classes, a modification of von Neumann's system". *Journal of Symbolic Logic*, no 2, p29-36.

[2] Throughout this book, the ordinals, denoted in current literature by the greek letters, will be denoted by the letters W and *w* supplemented later with numerical indices, W_1, or w_3, etc. In general, W or *w* designate a variable ordinal (any ordinal whatever). In particular, we employ the expressions "for every ordinal W". We use the notation with indices to designate a particular ordinal, as in the expression "take ordinal W_1 which is the matter of Number N_1". The subscripts will be used most often to the left of the sign (member), to designate an ordinal that is a member of another, as in writing w_1 (member) W (the ordinal w_1 is an element of the ordinal W).

ordinal.

8.6. If transitivity is a property of stability, this time we discover a complementary property of *homogeneity*: that which makes up the internal multiple of an ordinal, the elements that are members of it, are all ordinals. An ordinal is the count-for-one of a multiplicity of ordinals.

This homogenous and stable "fabric" of ordinal multiplicity leads me to say that ordinals are the ontological schema of the *natural* multiple. I call "natural" (by way of opposition to multiplicities that are unstable, heterogeneous, *historical*, and are consequently exposed to the evental caesura) precisely that which is exemplified by the underlying multiple-being, that which mathematics thinks: a maximal consistency, an immanent stability without lacuna, and a perfect homogeneity, in so far as that of which this multiple-being is composed is *of the same type* as itself.

We therefore propose once and for all that an ordinal is the index of the being of a natural multiplicity.

8.7. If it is true that the ordinals constitute the great ontological 'ground' of number, then we can also say that number is a figure of natural being, or that number *proceeds from Nature*. With the caveat however that "Nature" refers here to nothing sensible, to no experience: 'Nature' is an ontological category, a category of the thought of the pure multiple, or set theory.

8.8. Must we say at the same time that the ordinals "are numbers"? That would be very much the idea of Cantor, who thought to reach by way of the ordinals a prolongation into the infinite of the series of whole numbers. But for we who have not yet proposed any concept of number, this would be begging the question. We will see, after having defined what I call Number (the capitalisation not for the sake of majesty, but to designate a concept that subsumes all the species of number, known or unknown), that the ordinals, though playing a decisive role in this definition, are only the *representable* among numbers, in the numerical swarming which being lavishes *on the ground of Nature*. The ordinals will thus be at once the instrument of *our*

access to number, of our thinking of number, *and*, although lost in a profusion of Numbers that exceeds them in every way, they will be representable or figurable as being themselves, also, Numbers.

8.9. The empty set, 0, is an ordinal. That it is transitive, we have seen above (**8.1**). That all its elements are also transitive follows from this: it has *no* elements, how could it have one that *was not* transitive? Contrary to all intuition, zero, or the void, is a *natural* ontological given. The void, which sutures to being every language and every thought, is also the point of nature where number is anchored.

8.10. Von Neumann ordinals have two crucial properties:

1) They are totally ordered by the fundamental ontological relation, membership, the sign of multiple-presentation. That is to say that given two ordinals W_1 and W_2, either one is a member of the other ($W_1 \in W_2$), or the other way around ($W_2 \in W_1$), or they are identical ($W_1 = W_2$).

2) They obey a principal of minimality: given any property P whatsoever, *if* an ordinal possesses this property, *then* there exists a smallest ordinal to possess it. Order is always membership: if you have an ordinal W such that it possesses the property P (the statement P(W) is true), then there exists an ordinal W_1 which has the property and which is the smallest to have it (if $W_2 \in W_1$, W_2 *doesn't have* the property).

These two properties are natural. The first expresses the universal intrication of those stable and homogenous multiplicities that are natural multiplicities (*cf*. **8.6.**): thought in their being, two natural multiples – two ordinals, then – cannot be independent. Either one is in the presentation of the other, or vice versa. Nature does not tolerate indifference or disconnection. The second property expresses the 'atomic' or, if you like, 'quantum' character of nature. If a property applies to some natural multiple, then there is always a natural multiple that is the minimal support of that property,.

Taken together, these two properties articulate the global law of nature over its local law. Given that *a* nature does not exist (there is no set of all the ordinals, *cf*.

6.11.), there is a sort of unity of plan, of global interdependence, between natural multiples: the presentation of which they are the scheme is always "embedded". And, although there are not (unless one considers the void as such) unique and indiscernible components of nature like the atoms of the Ancients, there is locally a point of exception for every property that belongs to the "regions" of nature: the minimal support of this property.

This articulation of the global and the local gives its ontological framework to all of Physics.

8.11. The two crucial properties (total order and minimality) can both be *proved* on the basis of von Neumann's definition of the ordinals.

These proofs depend upon a key principle of set theory (of the ontology of the multiple): the axiom of foundation[3]. This axiom says that every situation (every pure multiple) comprises at least one term (one element) that has "nothing in common" with the situation, in the sense that nothing of that which composes the term (no element of the element) is presented in the situation (is a member of the original multiple).

8.12. Take again the example of my cat (*cf.* **7.6.**). It is an element of the set of living beings, and it is composed of cells that are in turn elements of this set, if one grants that they are living organisms. But, if we decompose a cell into molecules, then into atoms, we end up coming across purely physical elements, which are not members of the set of living beings. There is a term (perhaps the cell, in fact) which belongs to the set of living beings, but none of whose the elements belong to the set of living beings, because they all involve only 'inert' physicochemical materiality. Of this term, which is a member of the set but none of whose members belong to it, we can say that it grounds the set, or that it is a fundamental term of the set. 'Fundamental'

---

[3] The axiom of foundation, also called the axiom of regularity, was anticipated by Mirimanoff in 1917, and brought to full light by von Neumann in 1925. To begin with it was a matter, above all, of eliminating what Mirimanoff called "extraordinary sets", meaning those which are elements of themselves, or which contain an infinite chain of the type $... \in a_{n+1} \in a_n \in ... \in a_2 \in a_1 \in E$. It was afterwards realised that this axiom enabled a hierarchical presentation of the universe of sets.

For a historical and conceptual commentary on this axiom, cf. A Fraenkel, Y. Bar-Hillel and A. Levy, *Foundations of Set Theory*, North-Holland, 2nd ed., 1973, p.86-102.

means to say that on one side of the term, we break through that which it constitutes, we leave the original set, we exceed its preservative capacity.

8.13. Once more, leave for a moment living beings, cats, cells and atoms. Consider the singleton of the singleton of the void, that is the set whose unique element is the singleton of the void, and which we write as ((0)). The element (0) of this set itself has for its only element the void, 0. Now the void *is not* an element of the original set ((0)), whose only element is precisely (0), because the void 0 and the singleton of the void (0) are different sets. So (0) represents, in ((0)), a local point of foundation: it has no element in common with the original set ((0)). That which it presents *qua* multiple – that is, 0 – is not presented by ((0)), in the presentation of which it figures.

The axiom of foundation tells us that this situation is a law of being: every multiple is founded, every multiple comprises at least one element which presents nothing of that which the multiple itself presents.

8.14. The axiom of foundation has a remarkable consequence, which is that *no set can be a member of itself*, or that no multiple figures in its own presentation, or that no multiple counts itself as one. In this sense, *being knows nothing of reflection.*

In fact, take a set E which is an element of itself: one has $E \in E$. Consider the singleton of this set, (E). The only element of this singleton is E. So it must be that E founds (E). But this is impossible, since E is a member of E, and thus always has in common with (E) that element which is itself. Since the axiom of foundation is a law of being, we must reject the starting hypothesis: there does not exist any set that is an element of itself.

8.15. Returning to the crucial properties of the ordinals: One can prove them, as soon as one assumes the axiom of foundation. I will do so here for the principle of minimality. For the principal of total order according to membership, see the note[4].

For a philosophical commentary, *cf.* A Badiou, meditation 18 of *L'Etre et l'Evenement*, op.cit.

[4] A good presentation of the fact that membership ($\in$) orders the ordinals totally (strict order), in other words that given two different ordinals W_1 and W_2, one has either $W_1 \in W_2$ or $W_2 \in W_1$, can be found in J.R.Shoenfield, *Mathematical Logic*, Addison-Wesley, 1967, p.246-247.

Take an ordinal W_1 which has the property P. If it is minimal, all is well. Suppose that it is not. In that case, there exist ordinals smaller than W_1 (therefore, which are members of W_1, since the order considered is membership) and which have the property. Consider the set E of these ordinals (taking 'together' all those which have the property P and are members of W_1). The set E obeys the axiom of foundation. Then there is an element W_2 of E which is an ordinal (since E is a set of ordinals), which possesses the property P (since all the elements of E possess it), and which has no element in common with E.

But, since W_1 is an ordinal, it is transitive. So W_2, which is a member of it, is also a part of it: the elements of W_2 are all elements of W_1. If an element of W_2 has the property P, as it is an element of W_1, *it must be a member of E* (since E is the set of *all* the elements of W_1 which have property P). Which cannot be, because W_2 founds E and therefore has no element in common with E. In consequence, no element of W_2 has the property P, and W_2 is minimal for this property. Which was to be proved.

8.16. Thus is knitted the ontological fabric from which the numbers will be cut. Homogenous, intricate, originating from the void, locally minimisable for every property, it is very much what we could call a *horizonal structure*.

This demonstration is reproduced and commented upon in *L'Etre et l'Evenement*, *op.cit*., in paragraph 3 of meditation 12, p.153-158.

9. Succession and limit. The Infinite.

9.1. In chapter VI, when we spoke of Dedekind's and Cantor's approach to the notion of the ordinal (proceeding from well-orderedness), we saw that the whole problem was that after one ordinal comes another, well determined, and that this series can be pursued without end. We also saw that it was not at all the same thing to "pass" from *n* to *n*+1 (its successor) as to pass from "all" the natural numbers to their beyond which is the infinite ordinal ω. In the latter case, there is manifestly a shift, the punctuation of a "passage to the limit".

In the ontologised concept of the ordinals which von Neumann proposed and to which we dedicated chapter VIII, do we find once more this dialectic between simple succession and the 'leap' to the infinite? And more generally, in this new context, how does the thorny issue of the existence of an infinite multiple present itself?

9.2. Let us apply ourselves firstly to the concept of succession.

We must be careful here. The image of succession, of "passage" to the next, is so vividly present in the immediate representation of number that one often thinks that it is *constitutive* of its essence. I reproached J.A. Miller (*cf*. **3.17.**) precisely for reducing the problem of number to the determination of that which insists in its successoral engenderment. I held that the law of the serial traversing of the numeric domain, a law which *we* impose, does not coincide with the ontological immanence of number as singular form of the multiple.

Consequently, if we find the idea of succession once again in von Neumann's conception of the ordinals, it too must yield to the process of ontologisation. Our goal will be to discover, less a principle of traversal than an intrinsic qualification of *that which* succeeds, as opposed to that which does not. What counts for us is not succession, but the *being of a successor*. The repetitive monotony of Peano's *+1* does not concern us: what we want to think is the proper being of that which allows us to reach it only in the modality of additional steps.

9.3. Let us consider an ordinal W, in von Neumann's sense (a transitive of which all the elements are transitive).

A set, then, whose elements are:

– all elements of W;

– W itself.

So, we "add" to everything that composes the multiple W *one* additional element, namely W itself. It is a matter of the adjunction of a *new* element, since we know (it is a consequence of the axiom of foundation, *cf.* **8.14.**) that W is never an element of itself.

You can see a non-operative form of +1 emerging: it is not a matter of an extrinsic addition, of an exterior "plus", but of a sort of immanent torsion, which "completes" the interior multiple of W with the count-for-one of that multiple, a count whose name is precisely W. The +1 consists here in extending the rule of the assembly of sets to what had previously been the principle of this assembly, that is the unification of the set, W, which is thereafter aligned with its own elements, counting *with them*.

9.4. Let us give an example of the procedure.

We have demonstrated that the set D, which is written (0,(0)), and which is the pair of the void and the singleton of the void, is an ordinal (it is transitive and all its elements are transitive). Our non-operative definition of +1 consists of forming the set of the *three* following elements: the two elements of D, and D itself. We write this as (0,(0),(0,(0))) (the "whole" D is found in the third position). Call this triplet T. We can now demonstrate that:

– T is transitive. In fact, its first element, 0, is a universal part, so 0 is a part of T; its second element, (0), is the singleton of its first element, 0. So it is also a part of T (*cf.* **7.11.**). Its third element (0,(0)) is nothing but the 'putting together', the pair, of these first two. So it is also a part. Every element of T is a part, T is transitive.

– All the elements of T are transitive. Given that we have shown that D is an ordinal, we have duly shown that its elements, 0 and (0), are transitives. We have

equally demonstrated that it itself, (0,(0)), is transitive. And these are precisely the three elements of T.

So T, obtained in "adjoining" D to the elements of D, is a von Neumann ordinal: a transitive set of which all the elements are transitive.

9.5. The reasoning we are going to follow can easily be generalised. If W is any ordinal whatsoever, everything will follow just as for T: the set obtained in adjoining to the elements of W, W itself as an element, is an ordinal.

One "steps" from W to a new ordinal by adjoining to the elements of W a single additional element (this, now, allows us to lift a corner of the veil on the identity of our example T: in the same way that D was two – I would like to say *the being of number Two* – T is no other than the number Three).

The fact that one steps from W to a new ordinal, whose elements are those of W supplemented by the one-name of their assembly, by way of a sort of immanent +1, justifies the following definition: *we will call the successor of the ordinal* W*, and will denote* S(W)*, the ordinal obtained by joining* W *to the elements of* W.

So in our example, T (three) is the successor of D (two).

9.6. The idea of the 'passage' from two to three, or from W to S(W) is, in truth, a pure metaphor. In fact, from the start *there are* figures of a multiple-being, D and T, and what we have defined is a *relation* whose sole purpose is to facilitate *for us* the intelligible traversal of their existences. Finitude demands the binding of the un-binding of being. We therefore think, in the succession T = S(D), a relation whose basis is, in fact, immanent: T has the structural property, verifiable in its ontological composition, of being the successor of D, and it is no more than a necessary illusion to represent T as being constructed or defined by the relation S which connects it externally to D.

A more rigorous philosophical approach consists of examining the ordinals in themselves, and asking ourselves whether they possess the *property of succeeding*. For example, T has the property of succeeding D, *recognizable in itself from the fact that* D *is an element of* T, and what is more – as we shall see – an element which can

be distinguished in an immanent way (it is "maximal" in T).

We will call *ordinal successor* an ordinal that has the property of succeeding.

So T is an ordinal successor.

9.7. One might object that the property 'succeeds W' remains latent in the intrinsic concept of successor, and therefore that we have failed to establish ourselves in the ontological unbinding. We will answer this objection.

Let us consider an ordinal W which has the following purely immanent property: amongst the elements of W, there is *one* element, say w_1, of which *all the other* elements of W are elements: if w_2 is an element of W different from w_1, then $w_2 \in w_1$. I say that W is necessarily an ordinal successor (in fact, it succeeds w_1).

In fact, if this situation obtains, it is because W has as its elements:

– On one hand the element w_1;

– on the other the elements that, like w_2, are elements of w_1.

But in reality, *all* of the elements of w_1 are elements of W. Because we know that membership, $\in$, is a total order over the ordinals (*cf.* **8.10.**) Now all the elements of an ordinal are ordinals (*cf.* **8.5.**). Specifically, all the elements of W are ordinals. w_1 is therefore an ordinal, and it follows that the elements of w_1 are all ordinals. These elements are connected to ordinal w_1 and W by the relation of total order that is membership: if we have $w \in w_1$, since $w_1 \in W$, then $w \in W$ (transitivity of the relation of order).

Thus W is composed of all the elements of w_1, and w_1 itself: W is by definition the successor of w_1.

Let us agree to call the *maximal element* of an ordinal the element of that ordinal which is like w_1 for W: all the other elements of the ordinal are members of a maximal element. But the reasoning above permits us the following definition: *An ordinal will be called successor if it possesses a maximal element.*

Here we are in possession of a totally intrinsic definition of the ordinal successor. The singular existence of an "internal" maximum, located solely through the examination of the multiple structure of the ordinal, of the fabric of elementary membership at its heart, allows us to decide on its being a successor or not.

9.8. Since we now have an immanent, non-relational and non-serial concept of "what a successor is", we can pose the question: Are there ordinals that *are not* successors?

9.9. The empty set, 0, is an ordinal that is not a successor. It obviously cannot succeed anything, since it has no elements, and to succeed it must have at least one element, namely the ordinal that it succeeds.

Or, staying closer to the immanent characterisation: to be a successor, 0 must have a maximal element. Having no elements, it cannot be a successor.

Once again we discover the void's function as ontological anchor: purely decided in its being, it is not *inferable*, and, in particular, it does not succeed: the void is itself *on the edge of the void*, it cannot follow from being, of which it is the aboriginal point.

9.10. All the ordinals that we have used in our examples, and which are not the void, are successors. Thus (0) (which is the number 1) is the successor of 0. The number 2, whose being is (0,(0)), and which is composed of the void and 1, is the successor of 1. And our T (the number 3), which is composed of the void, 1, and 2, and is written (0,(0),(0,(0))), is the successor of 2. It is clear that we can continue, and will thereby obtain 4, 5, and finally any whole natural number whatsoever, *all the ordinal successors*.

9.11. Does this mean that we have at our disposal a thinking of the whole natural numbers? Not yet. We can say that 1, then 2, then 3, etc., if we think each in its multiple-being, are whole natural numbers. But, without being able to determine the *site* of their deployment, it is impossible for us to pass beyond this case-by-case designation, and to propose a general concept of whole number. As Dedekind saw, such a concept necessitates a detour through the infinite, since it is within the infinite that the finite insists. The only thing that we can say with certainty is that the whole numbers are ordinal successors. But this is certainly not sufficient to characterise them: there could well be other successors that were not whole numbers, even

successors that were not even finite sets.

9.12. The question becomes: are there any other non-successor ordinals *apart from the void*?

It is convenient (without knowing yet whether they exist) to call these non-successor ordinals different from 0 *limit ordinals*. We ask once more: do limit ordinals exist?

We are not yet in a position to decide upon this question. But we can prove that, *if they do*, they are structurally very different from successor ordinals.

9.13. No ordinal can come between an ordinal W and its successor S(W). By this we mean that, given that the relation of order between ordinals is membership, no ordinal W_1 exists such that we have the series $W \in W_1 \in S(W)$.

We know in fact that W is the maximal element in S(W) (*cf*. **9.7.**). In consequence, *every* element of S(W) that is different from W belongs to W. Now, our supposed W_1 is a member of S(W). Therefore, one of two things apply:

– Either W_1 is identical to W – But this is impossible, because we have supposed that $W \in W_1$, which would give us $W \in W$;

– Or W_1 is an element of W – but then it would not be possible to have $W \in W_1$, since one has $W_1 \in W$.

We see that ordinal succession is the scheme of 'one more step', understood as that which hollows out a void between the initial state and the final state. Between the ordinal W and its successor S(W), there is *nothing*. Meaning: nothing natural, no ordinal. We could also say that a successor ordinal delimits, just 'behind' itself, a gap where nothing can be established. In this sense, rather than succeeding, *a successor ordinal begins*: it has no attachment, no continuity, with that which precedes it. The successor ordinal opens up to thought a beginning in being.

9.14. It is entirely different with a limit ordinal, if such a thing exists. The definition of such an ordinal is, let us note, purely negative: it *is not* a successor, that is all that

we know of it for the moment. We can also say: it does not possess a maximal element. But the consequences of this lack are considerable.

Take L, a supposed limit ordinal, and w_1 an element of this ordinal. Since w_1 is not maximal, there certainly exists an element w_2 of L which is larger than it: so we have the chain: $w_1 \in w_2 \in$ L. But, since in its turn w_2 is not maximal, there exists a w_3 such that we have: $w_1 \in w_2 \in w_3 \in$ L. And so on.

Thus, whenever an ordinal is a member of a limit ordinal, a third is intercalated in the relation of membership, and, as this process has no stopping-point, as there is no maximal element, we can say that between any element w of a limit ordinal L and L itself, there is always an 'infinity', in the intuitive sense, of intermediate ordinals. So it is in a strong sense that the limit ordinal does not succeed. No ordinal is the last member of it, the "closest" to it. A limit ordinal is always equally "far" from all the ordinals that are members of it. Between the element w of L and L, there is an infinite distance where these intermediaries swarm.

The result is that, contrary to what is the case for a successor ordinal, a limit ordinal does not hollow out any empty space behind itself. No matter how 'close' to L you imagine an element w, the space between w and L is infinitely populated with ordinals. The limit ordinal L is therefore in a relation of *attachment* to that which precedes it; an infinity of ordinals "glues" it in place, stops up every possible gap.

If the successor ordinal is the ontological and natural schema of radical beginning, the limit ordinal is that of the *insensible result*, of transformation without gaps, of infinite continuity. Which is to say that every action, every will, finds itself either under the sign of the successor, or under the sign of the limit. Nature here furnishes us with the ontological substructure of the old problem of revolution (*tabula rasa*, empty space) and of reform (insensible, consensual and painless gradations).

9.15. There is another way to indicate the difference between successors and limits (which are for us the predicates of natural multiple-being).

Call *union of a set E* the set constituted by the *elements of the elements of E*. This concerns a very important operator of the ontology of the multiple, the operator of *dissemination*. The union of E "breaks open" the elements of E, and collects all the products of this breaking open, all the elements contained in the elements of which E

assures the count-for-one.

An example: Take our canonical example of three, the set T that makes a triplet of the void, the singleton of the void, and the pair of the void and its singleton. It is written (0,(0),(0,(0))). What is the union of T?

The first element of T is 0, which has no elements. It therefore donates no element to the union. The second element is (0), of which the one element is 0. This latter element will feature in the union. Finally the third element is (0,(0)), of which the two elements are 0 (which we already have) and (0). Ultimately, the union of T, set of the elements of its elements, is composed of 0 and (0): it is the pair (0,(0)). That is to say our D, or the number two. The dissemination of three is no other than two. Let us state in passing (this will be clarified in **9.18.**) that the union of T is "smaller" than T itself.

9.16. The position of the ordinals with regard to union is most peculiar. Given that an ordinal W is transitive, all its elements are also parts. The result is that *the elements of the elements of W, which are also the elements of the parts of W, are themselves elements of W*. In the union of an ordinal we find nothing but the elements of that ordinal. That is to say that the union of an ordinal *is a part of the ordinal*. If we denote the set "union of E" ∪E, we have, for every ordinal, ∪W ⊂ W.

This property is characteristically natural: the internal homogeneity of an ordinal is such that dissemination, breaking open that which it composes, never produces anything other than a part of itself. Dissemination, when one applies it to a natural multiple, delivers only a "piece" of that multiple. With regard to dissemination, nature, stable and homogenous, never "escapes" its proper constituents. Or: there is not in nature any non-natural *ground*.

9.17. That the union of an ordinal should be a part of that ordinal, or that the elements of elements should be elements, brings us to the question: are they *all*? Do we ultimately find not even a "partial" part (or *proper* part, *cf.* **4.12.**), but simply the ordinal we began with? It could well be that *every* element can be found as element of an element, since the internal fabric of an ordinal is totally intricated. In that case, one has ∪W=W. Not only does dissemination return only natural materials, but it

restores the initial totality. The dissemination of a natural set would be a *tautological* operation. Which is to say that it would be absolutely in vain: we could then conclude that *nature does not allow itself to be disseminated.*

9.18. This seductive thesis is verified *in the case of limit ordinals*, if such a case exists.

In fact take any element w_1 whatsoever of a limit ordinal L. We have shown (*cf*. **9.14.**) that between w_1 and L necessarily comes an intercalated element w_2, in such a fashion that we always have (whatever the element w_1) the chain $w_1 \in w_2 \in$ L. But also, when we disseminate L, the element w_1 will be found again in the union, as an element of w_2. In consequence, *every* element of L features in $\cup$L, the union of L. And as we have seen, inversely (*cf*. **9.15.**), that every element of $\cup$L is an element of L (since $\cup$L $\subset$ L), it only remains to conclude that the elements of L and those of $\cup$L are exactly the same. Which is to say that L is identical to $\cup$L.

Thus, to dissemination the limit ordinal opposes its infinite self-coalescence. It is exemplarily natural, insofar as in being "dissected" its elements do not alter. It *is* its own dissemination.

9.19. A successor ordinal, on the other hand, resists being identified with its dissemination. It remains *in excess* of its union.

Let us consider a successor ordinal W. It possesses by definition a maximal element w_1. Now it is impossible that this element should be found in the union of W. If it was so found, that would mean that it was the element of another element, w_2, of W, we would have $w_1 \in w_2$, and w_1 would not be maximal. The maximal element w_1 *necessarily makes the difference between W and $\cup$W*. There is at least one element of an ordinal successor that blocks the pure and simple disseminative restoration of its multiple-being. A successor, unlike a limit, is 'contracted', altered, by dissemination.

9.20. In my opinion this contrast is of very great philosophical importance. The prevailing idea is that what happens 'at the limit' is more complex, and also more obscure, than that which is in play in a succession, or in a simple 'one more step'. For

a long time, philosophical speculation has fostered a sacralisation of the limit. That which I have called elsewhere[1] the 'suture' of philosophy to the poem rests largely upon this sacralisation. The Heideggerian theme of the Open, of the deposition of a closure, is the modern form of the assumption of the limit as uprooting through counting, through technique, through the succession of discoveries, through the production of Reason. There is an *aura* of the limit, and an unbeing of succession. The "coming heart of another epoch" aspires (and this effect of the horizon can only be captured, it seems, by the poem) to a movement across "these endless meadows where all time stands still"[2].

What the ontology of the multiple (based on a contemporary Platonism) teaches us is, on the contrary, that the difficulty resides in succession, and that there also resides *resistance*. Every true test for thought originates in the localisable necessity of an additional step, of a unbreachable beginning, which is not *united* by the infinite filling up of that which precedes it, nor identical to its dissemination. To understand and endure the test of the additional step, this is the true necessity of time. The limit is a recapitulation of that which composes it, its "profundity" is fallacious, it is because of its not having any *gap* that the limit ordinal, or any multiplicity "at the limits", attracts the evocative and hollow power of such a "profundity". The empty space of the successor is more redoubtable, it is *truly* profound. There is nothing

---

[1] A. Badiou *Manifeste pour la philosophie*, Paris, éd. du Seuil, 1989. The circumstances and the effects of the suture of philosophy to the poem, beginning with Nietzsche and Heidegger, are described briefly in chapter VII, entitled "L'age des poetes".

[2] Ossip Mandelstam, [translated from the Russian by Ilya Shambat – Badiou quotes a French translation by Tatiana Roy – trans]:

Here is the discus, like a golden sun -
A blessed moment - in the air it stands -
The world is held in time like apple in one's hands -
Here will be heard only the Grecian tongue.

A solemn zenith of the service to God's will,
Light of round cupolas glows in July,
That with full chest, outside of time we sigh
Of endless meadows where all time stands still.

Like noon eternal is the Eucharist -
All drink the cups, all play and sing aloud,
Before the eyes of all the cup of God
Pours with a gaiety that can't desist.

The instant of Presence is beyond all insistence, all succession. The "eternal midday" is the transtemporal limit of time. Here is the conjoint site of the poem and the sacred.
It is not always in this place, we must say, that Mandelstams's poems establish themselves. Because his strongest poetry tries to think the century, and in this he succeeds.

more to think in the limit than in that which precedes it. But in the successor there is a crossing. The audacity of thought is not to repeat "to the limit" that which is already entirely held in the situation which the limit limits; the audacity of thought is to cross a space where nothing is given. We must learn once more how to succeed.

9.21. Basically, what is difficult in the limit is not what it gives us to think, but its *existence*. And what is difficult in succession is not its existence (as soon as the void is guaranteed, it follows ineluctably) but that which begins in thought with this existence.

Also, speaking of the limit ordinal, the question returns, always more insistent: do limit ordinals exist? On condition of the existence of the void, there is 1, and 2, and 3..., all successors. But a limit ordinal?

The reader will have realised: we find ourselves on the verge of the decision on the infinite. No hope of *proving* the existence of one single limit ordinal. We must make the great modern declaration: the infinite exists, and what is more it exists in a wholly banal sense, being neither revealed (religion), nor proved (mediaeval metaphysics), but being only decided by the injunction of being, in the form of number. All our preparation only adds up to saying, to being able to say, that the infinite can be thought *in the form of number*. We know this, at least for that which is the natural ontological horizon of number: the ordinals. That is infinite which, not being void, meanwhile does not succeed. It is time to announce the following:

Axiom of the infinite. **A limit ordinal exists**.

10. Recurrence, or Induction

10.1. A momentary pause to begin with: let us recapitulate that the ordinals prompt us to think being *qua* being, by way of a philosophy informed by mathematical ontology.

10.2. The ordinals are, because of the internal stability of their multiple-being (the maximal identity between membership and inclusion, between "first" presentation through the multiple, as element, and re-presentation through inclusion, as part) and the total homogeneity of their internal composition (every element of an ordinal is an ordinal), the ontological schema of *natural* multiplicity.

10.3. The ordinals do not constitute a set: no multiple form can totalise them. There exist pure natural multiples, but *Nature does not exist*. Or, to speak like Lacan: Nature is not-whole, just as being *qua* being is not, since there doesn't exist a set of all sets either.

10.4. The anchoring of the ordinals in being as such is twofold.

The absolutely initial point that assures the chain of ordinals through its being is the empty set 0, decided axiomatically as secularised form, or number-form, of Nothingness. This form is nothing other than the situational name of being *qua* being, the suture of every situation-being, and of every language, to its latent being. The empty set being an ordinal, and therefore a natural multiple, we can say the following: the point of being of every situation is natural. This statement is the foundation of materialism.

10.5. The limit-point that "relaunches" the existence of the ordinals beyond Greek number (the whole natural finite numbers; on Greek number, *cf.* chapter I) is the first infinite set, ω, decided axiomatically as a secularized form, and thus entirely subtracted from the One, from infinite multiplicity.

From this point of view, the ordinals represent the *modern* scale of

measurement (conforming to the two crucial decisions of modern thought) of natural multiplicity. They *tell us* that nothingness is a form of natural and numerable being, and that the infinite, far from being found in the One of a God, is omnipresent in nature, and what is more in every existing-situation.

10.6. *Our* traversal (or the limits of our representation) of the ordinals arranges them according to an untotalisable series. This series 'starts' with 0. It continues through the whole natural numbers (1,2,...,n,n+1,..., etc.), numbers whose form of being *is composed of the void* (under the forms (0),(0,(0)),(0,(0),(0,(0))), ..., etc.). It is continued by an infinite (re)commencement, guaranteed by the axiom "there exists a limit ordinal" which authorises the inscription, beyond the series of whole natural numbers, of ω, the first infinite ordinal. This recommencement opens a new series of successions: ω, ω+1, ..., ω+n, ...,etc. This series is closed beyond itself by a *second* limit ordinal, $\omega + \omega$, which inaugurates a new series of successions, etc. One thus has the representation of a series of ordinals, deployed with no conceivable stopping-point, which moves within the infinite (beyond ω) just as in the finite.

10.7. The ordering principle of this series is in fact membership itself: given two ordinals W_1 and W_2, we have $W_1 \in W_2$, *or* $W_2 \in W_1$, *or* $W_1 = W_2$. Membership, a unique ontological relation because it regulates the thinking of multiple-being as such, is also that which totally orders the series of ordinals. So that if W is an ordinal and S(W) its successor, one will have $W \in S(W)$. So that, if n is a whole natural number (a finite ordinal) and n' a "bigger" whole number, we have $n \in n'$. So that, for any whole natural number n whatsoever, we have $n \in \omega$ (the first infinite ordinal), etc.

10.8. There are three types of ordinal (given the modern decisions which *impose* the void and the infinite):

- The empty set, 0, is the inaugural point of being;

- The successor ordinals adjoin to their predecessor *one* element, namely that predecessor itself. We call S(W) the successor of W. W is the maximal element in S(W), and the presence of a maximal element allows us to characterise

successors in a purely immanent (non-serial) fashion. The successor ordinals give us a numerical scheme of that which says 'one more step'. This step consists always in supplementing all that one has with a unique mark for that 'all'. To take 'one more step' is the same as making one of a given multiplicity, and adjoining that one to it. The new situation is "maximalised": it contains a term which dominates all the others;

-   The limit ordinals do not possess any internal maximal element. They mark the beyond proper to a series without stopping-point. They do not succeed *any* specific ordinal, but we can say that they succeed *all* the ordinals of the series of which they are the limit. No ordinal of this series is "closer" to the limit ordinal than any other. Because a third ordinal, and ultimately an "infinity" (in the intuitive sense of a series without stopping-point) of ordinals, come to be intercalated (according to the relation of order, which is membership) between every ordinal of the series and the limit ordinal. The limit ordinal adheres to everything that precedes it. This is specifically indicated by its identity with its dissemination (L = ∪L). The limit *totalises* the series, but does not *distinguish* any specific ordinal within it.

10.9. Just as a limit ordinal is structurally different from an ordinal successor (with regard to the internal maximum as with regard to dissemination), so the 'passage to the limit' is an operation of thought entirely different from 'take one more step'.

Succession is in general a *more difficult* local operation than the global operation of passage to the limit. Succession gives us more to think about than the limit. The widespread view to the contrary stems from the fact that, not yet being 'absolutely modern', we still tend to sacralise the infinite and the limit, which is to say: retain them still in the form of the One. A secularised thought, subtracted from the One and the sacred, recognises that the most redoubtable problems are local problems, problems of the type: ; 'How to succeed?', 'How to take one more step?'.

10.10. The space of the ordinals allows us to *define* the infinite and the finite. An ordinal is finite if, in the chain of order that regulates membership, it comes before ω. It is infinite if it comes after ω (including ω itself).

We can state that, conforming to Dedekind's intuition, only the existence of an infinite ordinal permits us to define the finite. Modern thought says that the first, and banal, situation is the infinite. The finite is a secondary situation, very special, very singular, extremely rare. The obsession with "finitude" is a remnant of the tyranny of the sacred. The "death of God" does not deliver us to finitude, but to the omnipresent infinity of situations, and correlatively, to the infinity of the thinkable.

10.11. The complete synthetic recapitulation of the fact that the ordinals give us to think being *qua* being, in its natural proposition, is now complete. Now we must turn towards *our* capacity of traversing and of rational mastery of this donation of being. One way is simply to proceed, in this endless fabric, to the carving-out of Number.

10.12. It is a blessing for our subjective finitude that the authority – properly without measure – of natural multiplicities allows that diagonal of traversal, or of judgment, which is reasoning by recurrence, also called complete induction, and, in the case of infinite ordinals, "transfinite induction". In fact this alone allows us to attain, in treating of an infinite domain (and even, if we consider the ordinals, one that is infinitely infinite), the *moment of conclusion*.

Supposing that we want to prove that *all* the ordinals possess a certain property P. Or that we want to establish rationally, by way of demonstration, a universal statement of the type: 'For all x, if x is an ordinal, then P(x)'. How can this be done? It is certainly impossible to verify case by case that it is so: the task would be infinitely infinite. Neither is it possible to consider the 'set of ordinals', since such a set does not exist. The 'all of the ordinals', that which is implied in the universal quantifier of the statement 'for all x', cannot be converted into 'all the elements of the set of ordinals'. Such a set is inconsistent (*cf.* **6.11.**). The lifting of this impasse is the role of reasoning by recurrence.

10.13. Reasoning by recurrence combines *a* verification and the demonstration of *an* implication. Once in possession of these two moments, the proper structure of the ordinals authorises the universal conclusion.

Let P be the property. We begin by verifying that the empty set 0 possesses

this property, we test P in the "case" of 0. If the empty set does not possess the property P, it is pointless to pursue the investigation. Since one ordinal, 0, does not have this property P, it is certainly false that all of the ordinals have this property. Suppose then that the statement P(0) is true, that the test in the case of 0 is positive.

We will now try to prove the following implication: *if* all the ordinals that precede some ordinal W (according to the total ordering of the ordinals, which is membership) have the property P, *then* W also has it.

Note that this implication does not say that an ordinal with the property P *exists*. It remains in the hypothetical register, according to the general type: 'if *x* is so, then that which follows *x* is so'. In reality, the implication is universal, it does not specify any ordinal W. It says only that, for every ordinal W, if one supposes that those which come before it in the chain of ordinals verify P, then one is constrained to admit that W verifies it also.

It is most often necessary to divide this demonstration (supposing that it is possible, which obviously depends on the property P), by treating separately the case where W is supposed to be a successor from the case where it is supposed to be a limit (since W is any ordinal whatsoever, it could be one or the other). Reasoning by recurrence, as we saw in the central implication that constitutes it, strongly binds that which is the case for an ordinal W to that which is that case for the ordinals that precede it. Now the relationship of a limit ordinal to the anterior ordinals (constituted by an infinite adherence) differs radically from that of a successor (which, between itself and its predecessor, clears an empty space). Because of this, the procedures of thought and of proof put into play in the two cases are usually quite heterogeneous. And, as the philosophy of this heterogeneity allows us to foresee (*cf*. **9.19.**), it is the case of the successor which is regularly found to be the most difficult.

Let us assume that we have verified the truth of P(0), and that we have proved the implication 'if for every ordinal *w* that precedes *w* (which is a member of W: order is membership), we have P(*w*), then we also have P(W)'. We can conclude that *all* the ordinals satisfy P, in spite of the fact that this "all" not only alludes to an infinitely infinite immensity of multiples, but that all the same it does not make an All. It is truly the infinite and inconsistency 'conquered word by word'.

10.14. What authorises such a passage to "all", such an adventurous "moment of conclusion"? The authorisation comes to us from a fundamental property of the ordinals as ontological schema of the natural multiple: their "atomistic" character, the existence, for every property P, as soon as one ordinal possesses it, of a *minimal* support for this property (*cf.* **8.10.** and **8.15.**).

If the conclusion were false – if it were not correct that all ordinals have the property P – that would mean that there would be at least one ordinal which did not have the property P. This ordinal would then have the property not-P, not-P meaning simply 'not having the property P, being a non-P'.

But, if there exists an ordinal that has the property not-P, *there exists a smaller ordinal which has this property not-P*, in virtue of the atomistic principle, or principle of minimality. And since it is the smallest to have the property not-P, all those which are smaller than it must have the property P.

We could object: these ordinals 'smaller than it' may not exist, because it is possible that the minimal ordinal for the property not-P could be the void, which is not preceded by anything. But no! Since (first moment of our procedure) we have verified precisely that 0 has the property P, the minimal ordinal for not-P cannot be 0. Thus it makes sense to speak of ordinals smaller than it; they exist, and must all have the property P.

Now our central implication, supposed proved, said exactly that, if all the ordinals smaller than a given ordinal have the property P, then that ordinal also has it. We have reached a formal contradiction: that the supposed minimal not-P must be a P. It is necessary then to conclude that this latter does not exist and that therefore all the ordinals do have the property P.

So the ontological substructure of natural mutiplicities comes to found the legitimacy of recurrence. Our verification (the case of 0) and our demonstration (if P(w) for all w such that $w \in$ W, then P(W) also), if it is possible (which depends on P...and on our mathematical knowhow), authorises the conclusion for 'all the ordinals'.

10.15. We have remarked, in studying Peano's axiomatic (*cf.* **5.3.**) that reasoning by recurrence is a fundamental given of *serial* numericality, of which the whole natural

numbers are an example. It is quite natural that it should extend to that 'universal series' which makes up the ordinals. But the great difference is that, founded in being (in the theory of the pure multiple), the principle of induction or of recurrence, rather than being, as in Peano, an axiomatic form or a formal disposition, is here a *theorem* – that is, a property *deducible* from the ordinals.

It is of the essence of the natural multiple, which escapes all totalising thought, to submit itself nonetheless to that intellectual 'grasp' which is the inductive schema. Here once more, being is found to be available to thought in that form of Number which is the conclusion for 'all' proceeding from and out of the verification for one only (here, 0), and of a general procedure which transfers the property of what comes 'before' (predecessor or series without end, depending whether the case is an successor ordinal or an limit ordinal) to what comes 'after'. Number is that which bestows being upon thought, in spite of the irremediable excess of the one over the other.

10.16. Reasoning by recurrence is a procedure of proof for universal statements concerning ordinals. It allows us to conclude. But there is a more important usage of recurrence, or of transfinite induction, one which allows us to *arrive at the concept.* This is *inductive definition.*

Suppose that the aim of our thinking is not to prove that this or that multiple, for example the ordinals, have a property P, but to define a property P, in a way that would allow us *then* to test it on multiples. A well-known difficulty in such a case is that *we do not know in advance whether a property defined in language is 'applicable' to a pure multiple without inconsistency.* We have seen, for example (*cf.* **2.11.**) that the property "is not an element of itself" does not apply to any existing set, and that its perfect formal correctness does not change the fact that, handled without care, it leads to the ruin, by way of inconsistency, of all formal thought. But how can we introduce limitations and guarantees, if language alone cannot support them? The procedure of definition by recurrence, or inductive definition, answers this question.

10.17. What will found the legitimacy of the procedure this time is that with the ordinals we have at our disposal a sort of universal scale, which allows us to define

the property P *at successive levels*, without exposing ourselves to the risk of inconsistency that envelopes the supposition of an All. Inductive definition is a *ramification* of the concept: the property P would not be defined "in general", but always by indexation to a certain level, and the operators of this indexation would be the ordinals. Here once again, being comes to the aid of finitude, in assuring for our thought that although the domain of being as pure multiple exceeds all parts, it can proceed by stages and fragments.

10.18. In conformity with the typology of ordinals, which distinguishes three types (the void, successors, limits), our procedure is divided into three.

- We first define *explicitly*, with a discursive statement, level 0 of the property. An explicit definition assumes that we have a property – let us say Q – *already* defined, and that we can affirm that level 0 of P – let us say P_0 – *is equivalent* to Q. We have: $P_0(x) \leftrightarrow Q(x)$.

- We then say that *if* level w of P is defined, P_w, *then* level S(w), that is, $P_{S(w)}$, is defined through an explicit procedure which we shall indicate. To say that P_w is defined is to say that there is a property – call it R – already defined such that P_w is equivalent to it, so $P_w(x) \leftrightarrow R(x)$. The existence of an explicit procedure enabling us to pass from the definition of P_w to that of $P_{S(w)}$ means that there is a function f that assures the passage of R (which defines P_w) to a property $f(R)$ which will define $P_{S(w)}$. Finally, we can say that 'x has the property $P_{S(w)}$' means 'x has the property $f(R)$', or that f, which permits the 'passage' from the definition of P_w to that of $P_{S(w)}$, is an explicit operation on R, stated once and for all.

- We will ultimately say that, *if* all the levels of P below an limit ordinal L have been defined, say: $P_0, P_1, ..., P_n, P_{n+1}, ...$, *then* the level L of P, say for example P_w, is defined by a "recollection" explicable by that which defines all the levels anterior to it (in this process, the union or dissemination in general plays a decisive role, for reasons we have given in **9.17.**). Most often we have something like: for a given x, $P_L(x)$ is true, if there exists a level below L, call it w, where $w \in L$, such that $P_w(x)$ is true. The limit level will assume, in conformity with its essence, all inferior levels, and will not introduce anything new.

Thus we will have at our disposal not just a single concept P, but an infinite

and infinitely-ramified family of concepts, from P_0, explicitly defined, up to the more considerable ordinal indexations P_w, through to $P_n, P_\omega, P_{\omega+n}$, etc. We will then be able to say that the concept P, as unique concept, is *defined by transfinite induction*, in the following sense: for a given x, $P(x)$ will be true if and only if there exists an ordinal W such that x possesses the property at level W. We would have the following equivalence: $P(x) \leftrightarrow$ 'there exists W such that $P_W(x)$'.

So the inductive mastering of the concept takes place by means of its ordinal ramification, and by means of the equivalence of 'the concept P holds for x' and 'the concept P holds for *x at level W of that concept*'. This equivalence *avoids all mention of the All*. It tests the property P, not 'in general', but on *one* level, which leaves it free from the paradoxes of inconsistency.

10.19. I will give an example of great interest, at once intrinsic (it sheds a keen light on the general structure of the theory of the pure multiple, or ontology: it proves that, thought in their being *qua* being, multiples are stratified) and methodological (we will see clearly the functioning of levels in the definition of the concept).

The underlying idea is to define, for each multiple, an ontological *rank*, indexed on the ordinals, which measures its "distance" in a certain sense, from that initial suture which is the empty set. We could also say that the rank is a measure of the *complexity* of a set, of the immanent intrication of the instances of the void that constitute it.

Naturally, it is impossible to speak of "all" the sets, to do that it would be necessary to collect them as the elements of a set of all sets, which would be inconsistent. The prudent gradual approach of the inductive procedure is indispensable here.

The two important operations which, in set theory, allow one to 'step' from one set to another are:

-   Union, or the set of elements of elements of the initial set. The operation of dissemination, which we have already met (*cf.* **9.15**). Given a set E, one denotes its union $\cup E$;

-   The set of parts, which consists of 'putting together' to make one all the *parts* of

the initial set, all that is included in that set (on membership and inclusion, *cf.* **7.3**). We denote p(E) the set of the parts of E. Note that the *elements* of p(E) are the *parts* of E, that to say e $\in$ p(E) is to say e $\subset$ E.

We will construct the hierarchy of ranks by means of these two operations. The property we will try to define through transfinite induction, according to the method explained in **10.18**, will be denoted R(x), to be read as : 'x possesses a rank' (or : 'x is well-founded'). Our three steps will be as follows:

1) *Explicit definition of the property of level 0.* We propose that $R_0(x)$ is not true for any x, in other words that $R_0(x)$ is equivalent to $x \in 0$.

2) *Uniform treatment of successive levels.* We propose that $R_{S(w)}(x)$ is true if and only if x is a member of the set of parts of the set constituted by all the z which satisfy R_w. In other words, the rank of the successor level S(w) is the set of parts of the rank defined for the level which indexes the predecessor w. We can write this as follows: $R_{S(w)}(x) \leftrightarrow ((y \in x) \rightarrow R_w(y))$: if x satisfies $R_{S(w)}$, the elements of x satisfy R_w, and in consequence x is a part of the set of sets which satisfy R_w. One can also write, denoting by R_w the set of x such that $R_w(x)$ is true:

$$(x \in R_{S(w)}) \leftrightarrow (x \subset R_w) \leftrightarrow (x \in p(R_w))$$

3) *Uniform treatment of limit levels.* As one would expect, it is union that is at work here. We say that $R_L(x)$ is true if x is in a rank whose index is smaller than L, that is if there exists a $w \in L$ such that $R_w(x)$ is true. Thus the rank R_L recollects all the elements of the ranks below it; it is the union of these ranks. With the same conventions as above, we can write: $(x \in R_L) \leftrightarrow x \in \cup R_w$ for all w smaller than L.

The property R is thereby wholly defined by induction. We can say that x possesses a rank, or that R(x) (without index) is true, if an ordinal w (successor or limit) exists such that $R_w(x)$ is true. This property "symbolises" that one arrives at the complexity of x, beginning from 0 (which defines the level R_0 of the property), through the successive usage of union and of passage to the parts, a usage whose "length" is measurable by an ordinal: the smallest ordinal w such that $R_w(x)$ is true.

10.20. That this procedure really "works", that it makes sense ultimately to speak of *the* property P, meanwhile, is not self-evident. The generosity of natural being consists in the fact that one can *prove* the effective character of this ramified determination of the concept[1].

Thus thought proceeds in the traversal of being, under the universal intricated and hierarchised rule of Nature, which doesn't exist, but prodigally provides measurable steps. Number is accessible to us through the law of such a traversal, at the same time as it sets the conditions, as we saw with the ordinals, for this traversal itself. Number is that through which being *organises* thought.

---

[1] For the demonstration of the validity of induction definitions, you are referred to a book by K.J.Devlin, *Fundamentals of Contemporary Set Theory*, Springer-Verlag, 1980, p65-70 ("the recursion principle")

11. The Whole Natural Numbers

11.1. The ordinals lead us directly to the Greek numbers: the whole natural numbers. We are even in a position to attach a new, non-Greek, legitimacy to the adjective "natural" which mathematicians, with the symptomatic subtlety of their appellations, adjoin to the civil status of these numbers: they are "naturals", by virtue of the fact that, in the end, they coincide purely and simply with the ordinals, which are the ontological schema of the pure natural multiple.

It is in effect "natural" to identify in its being the *site of number* (understood as: of whole number), a site whose existence Dedekind vainly tried to guarantee on the basis of the consideration of "all the possible objects of my thought", with the first infinite ordinal, ω, whose existence we decide with the modern injunction of being, by declaring the axiom "a limit ordinal exists".

11.2. To say that ω is the site of whole number has a precise set-theoretical meaning: that which "occupies" the site is that which is a member of it. Now not only are all the ordinals that precede an ordinal members of it, but they constitute *all* the elements of the initial ordinal.

In fact, we know that total order in the ordinals is really membership (*cf.* **8.10.**). In consequence an ordinal smaller than a given ordinal W is precisely an ordinal that is a member of W. The image of an ordinal (for example, one larger than *w*) is as follows:

$$0 \in 1 \in 2 \in ... \in n \in n+1 \in ... \in \omega \in \omega+1 \in ... \in W$$

where all the numbers in the chain of membership constitute exactly the elements of W. Visualised in this way, the ordinal W appears as a series of "embedded" ordinals, whose "length" is exactly W. There are in effect W links in the chain to arrive at W. We can also see an ordinal W, which contains exactly W ordinals (all those which precede it) as *the number of that of which it is the name*. Which is another way of saying that it is identified with the site where its predecessors insist, being the recollection of that insistence.

Thus the definition of whole natural numbers is entirely clear: an ordinal is a

whole natural number if it is an element of the first limit ordinal ω. The structure of the site of number is in this case:

$$0 \in 1 \in 2 \in ... \in n \in n \in n{+}1 \in ... \in \omega$$

But one must take care to note that ω itself, which is the name of the site, *is not a part of it*, since no set is a member of itself (*cf.* **8.14.**). The site of whole number, ω, is not an element of the site, *is not a whole number*. As ω is the *first* limit ordinal, it follows that all the whole numbers, except naturally the empty set 0, are successors.

11.3. An attentive reader might make the following objection: I say that ω is the first limit ordinal. But am I sure that a "first" limit ordinal exists? The axiom of infinity (*cf.* **9.20.**) says only: "*a* limit ordinal exists", it does not specify that this ordinal is "the first". What authorises our calling ω the "first limit ordinal", or first infinite ordinal? It could well be that as soon as I state: "a limit ordinal exists", a multitude of them appear, none of which is "first". There could be an infinite descending series of negative numbers, of which it is easy to see that there would be no first term: no whole negative number is "the smallest", just as no whole positive number is "the largest" (this second point is equivalent to saying that ω, the beyond and the site of the series of positive numbers, is a limit ordinal).

But if I cannot state, and determine in a unequivocal fashion, the first limit ordinal, then what becomes of my definition of whole numbers?

11.4. We can overcome this objection, once more thanks to that great principle of natural multiples that is minimality. We know that given a property P, *if* an ordinal exists which possesses that property, *then* there is one and only one minimal ordinal that possesses it. Take the property "is a limit ordinal ". There certainly exists an ordinal that possesses it, since the axiom of the infinite says precisely that. Thus, there exists one and only one limit ordinal that is minimal for this property. Consequently we can speak without hesitation of a 'first limit ordinal', or of the 'smallest limit ordinal', and it is to this unique ordinal that we give the proper name ω. There is therefore no ambiguity in our definition of whole natural numbers.

11.5. We must never lose sight of the fact that notations of the type 1,2,*n*, etc. are *ciphers*, in the sense of codes, which serve to designate multiples fabricated from the void alone. We have known for a long time (*cf*. already in **8.3.**) that 1 is in reality the singleton of the void, that is (0), that two is the pair of the void and the singleton of the void, that is (0,(0)), that three is the triplet of the void, the singleton of the void, and the pair of the void and singleton of the void, that is (0,(0),(0,(0))), etc. To further exhibit this weaving of the void with itself, let us write also the real being of the cipher 4: (0,(0),(0,(0)),(0,(0),(0,(0)))).

It is evident to us that 4 is a set of four elements, in the order 0, then (0), then (0,(0)), then (0,(0),(0,(0))). These four elements are none other than zero, 1, 2, and 3. The elements of a whole number comprise precisely all those numbers which precede it, which is not surprising since we have shown above that this is the innermost structure of every ordinal (*cf.***11.2.**). We could write: 4 =(0,1,2,3). And, as we have remarked, to pass from 3 to 4 (as from any *n* to *n*+1), one 'adjoins' to the elements of 3 (or of *n*) the number 3 itself (or the number *n*). Which is not surprising, since this is the general definition of succession in the ordinals (*cf*. **9.6.**).

It would obviously be impossible to use the procedure of succession to 'step' from some whole *n*, no matter how large, to the first limit ordinal ω. This is because ω, let us repeat, is not a whole number, it is the site of such numbers. An important law of thought emerges here (one which, we might say in passing, the Hegelian figure of Absolute Knowledge, supposed to be the "last" figure of Consciousness, contravenes), which states that *the site of succession does not itself succeed*.

11.6. Once we have at our disposal the site of the whole natural numbers, their multiple-being which fabricates in the finite the void alone, and the law of succession as law of *our* traversal of these numbers, we "rediscover" the classical operations (addition and multiplication for example) through simple technical manipulations, which arise from the general principles of inductive definition, or definition by recurrence, which we have explained and legitimated on the basis of natural being, in chapter X. It is time to give a new example.

11.7. Take a given number, say for example 4. We want to define by induction a function F whose meaning will be: for any number *n* whatsoever (therefore for *every* whole number, and there are an infinity of them), F(*n*) is equal to the sum 4+*n*. To achieve this we have at our disposal only one operator: ordinal succession, since the only thing we know is that all the whole numbers except 0 are successors. We will proceed exactly according to the schema explained in **10.18**, with the exception that we do not have to worry about the case of limit ordinals (since there is not one before ω). We will as always use S(*n*) to denote the ordinal successor of the whole number *n*.

- We first state : F(0) = 4 (entirely explicit value, of which the underlying intuition is that 4+0 = 4).

- then we proceed to the successoral induction by positing: F(S(*n*)) = S(F(*n*)). A regulated and uniform relation between the value of the function for S(*n*) and its value for *n*, a relation that uses only what we already know; the operation of succession, defined in general on the ordinals. The underlying intuition is that 4+(*n*+1) = (4+*n*)+1, to return to the usual "calculating" notation, which denotes the successor of *n* as *n*+1.

The value of the function is defined entirely by these two equations. Say for example I wanted to calculate F(2). I have the following mechanical steps :

$$F(0) = 4$$

$$F(1) = (F(S(0)) = (S(F(0)) = S(4) = 5$$

$$F(2) = (F(S(1)) = (S(F(1)) = S(5) = 6$$

We can see clearly that such a schema is a true *definition of addition*, through the use of recurrence, proceeding from the operation of succession alone. We can define multiplication in the same way, once we have obtained a general inductive schema of addition. Take P(*n*) the function to be defined, of which the value is *n* multiplied by 4. We begin the induction this time with 1 and not with 0, and state that, if F(*n*) is like our previous example (defined inductively as 4+*n*):

$$P(1) = 4 \text{ (guiding intuition : } 4 \times 1 = 4)$$

$$P(S(n)) = F(P(n)) \text{ (guiding intuition : } 4 \times (n + 1) = 4 + (4 \times n))$$

These technical manouevres are of no direct interest. They serve only to convince us that the whole numbers thought in their being (ordinals that precede ω,

fabricated from finite combinations of the void) are *also* the same ones with which we count and recount, as the epoch prescribes us to, without respite.

11.8. Thus is achieved the philosophico-mathematical reconstruction of whole numbers. They do not derive from the concept (Frege), nor can their site be inferred from our possible thoughts (Dedekind), nor is their law limited to that of an arbitrarily axiomatised operative field (Peano). They are, rather, in the retroaction of a decision on the infinite, that which in number proffers being in its *natural* and *finite* figure.

The whole numbers are Nature itself, in so far as it is exposed to thought, to the limited extent of its *capacity for finitude*. Again this exposition is possible only on condition of a point of infinity, the limit ordinal ω, the existential guarantee of whole number. This point of infinity is immense in relation to the whole numbers, since, underpinned by successoral repetition, it constitutes the site of their total exercise, a site without internal limits (succession can *always* continue). It is however tiny in relation to the profusion of natural infinite being beyond its first term ω. Whole number is the form of being of a finite 'nearly nothing', which being *qua* being deploys between the void and the first infinity.

11.9. It is but in anticipation without solid foundation, and in homage to their antiquity, that we call the whole naturals "numbers". We have already remarked (*cf.* **8.8.**) that, still without a general concept of number at our disposal, it would be illegitimate to say that the ordinals were numbers. Now, the whole numbers are none other than the ordinals. And number, or rather Number, qualifies a type of being of the pure multiple which exceeds the ordinals. Until we have made sense of this type, in such a way that it becomes applicable to all species of number (whole, relative, rational, real, ordinal, cardinal), we can only speak of "number", in a sense still insufficiently free of its operative intuition, or of the historical heredity of this signifier.

But our preparations are complete. The homage rendered to the Greek numbers was only the first step of a vast introduction, genealogical and then conceptual. It is now necessary to *define* Number.